Scaling for the Lorentz force: particle orbits

Just like we did for the generalized Ohm's law, we can scale the Lorentz force law to see what the "natural scaling" of the equations are. A problem with simulations is that computers work better with numbers like 10 and 0.1. Computers don't like dealing with big numbers like our particle density (10^{21} in MKS) or ion speed (10^{5} in MKS) at the same time as small numbers like the proton mass (10^{-27} in MKS) and proton charge (10^{-19} in MKS). Dimensionalizing an equation also gives a very good intuitive sense of what the equation means (we'll see that here). It also makes it easier to compare different systems (say SSX with ion orbits of cm and the solar wind with ion orbits of km).

First, here's the Lorentz force equation for charged particle orbits in arbitrary electric and magnetic fields. We will eventually want to explicitly pull out the charge-to-mass dependence:

$$m\frac{d\mathbf{v}}{dt} = q\left[\mathbf{E} + \mathbf{v} \times \mathbf{B}\right]$$

Since this is an MKS formula, every term is a force with units of newtons. The first step is to write the equation in dimensionless form. This is really important for doing numerical modeling.

Instead of MKS units, let's consider units more appropriate for SSX. Instead of seconds, we'll use a more typical time scale for the problem: $\tau = 10^{-6}\ s$ or something (we'll see a natural scale emerge later). We'll re-write every time $t \to \tau\tilde{t}$ where t was a time measured in seconds, $\tilde{t}$ is a dimensionless variable (that hopefully runs from 0.1 to 10 or something, and $\tau = 10^{-6}\ s$. This means that if $\tilde{t} = 1$ we're talking about a time step of a microsecond.

We do that for all the dynamical variables:

$$t \to \tau\tilde{t}$$

$$v \to v_{th,i}\tilde{v}$$

$$B \to B_0\tilde{B}$$

$$E \to E_0\tilde{E} = v_{th,i}B_0\tilde{E}$$

$$m \to Mm_p\tilde{m}$$

$$q \to Ze\tilde{q}$$

So for example, if $\tilde{v} = 0.1$ that means a velocity 1/10 the ion thermal velocity where $v_{th,i} = 10^4 m/s$ for 1 eV protons and $10^5 m/s$ for 100 eV protons. Similarly, $B_0 = 0.1T$ for us so $E_0 = 10^4 V/m$ is our natural electric field unit. Note that to be completely general, I left $\tilde{m}$ and $\tilde{q}$ as variables but by extracting the Z/M dependence, those dimensionless "variables" will be fixed at unity. So let's re-write the equation of motion:

$$Mm_p\tilde{m}\frac{d(v_{th,i}\tilde{v})}{d(\tau\tilde{t})} = Ze\tilde{q}\left[v_{th,i}B_0\tilde{E} + v_{th,i}\tilde{v} \times B_0\tilde{B}\right]$$

Note that there's a factor of $v_{th,i}B_0$ on the RHS. I'm going to pull everything to LHS to see what we have:

$$\left(\frac{Mm_pv_{th,i}}{\tau}\frac{1}{Ze}\frac{1}{v_{th,i}B_0}\right)\tilde{m}\frac{d\tilde{\mathbf{v}}}{d\tilde{t}} = \tilde{q}\left[\tilde{\mathbf{E}} + \tilde{\mathbf{v}} \times \tilde{\mathbf{B}}\right]$$

At this point, we have some choices. We could choose for our time scale the time it takes a thermal ion to go across SSX. This would mean $\tau = \ell/v_{th,i}$ (about 10 μs for us). This would also mean introducing a new scale to our equation (ℓ) which is inelegant. We notice that there's a factor of the proton gyro-frequency $\omega_{ci} = eB_0/m_p$ already in the pre-factor. If we choose for our time scale the proton gyro-period $\tau = \omega_{ci}^{-1} = m_p/eB_0$, then lots of things cancel nicely and we get a nice final equation:

$$m\frac{d\mathbf{v}}{dt} = \frac{Z}{M}q\left[\mathbf{E} + \mathbf{v} \times \mathbf{B}\right]$$

Dimensionless equation: Note that while this looks like the equation I initially wrote down, its very different. First, I dropped the tildes but each variable in there is dimensionless with a previously determined scaling. Second, had I gone with $\tau = \ell/v_{th,i}$, there would have been an extra factor of ρ_i/ℓ floating around (inelegant). Third, as noted above, m and q are really just fixed at unity and could be dropped. The dynamics of different ions is in the Z/M term.

So how does this work? If we pick a 1 eV proton in a 0.1 T field (SSX case), then an initial value of $v = 1$ corresponds to units of about 10^4 m/s. A time of $t = 1$ corresponds to a proton orbit time in a 0.1 T field (about 1 μs). An electric field of $E = 1$ would correspond to $E = vB_0 = 10^3$ V/m (a bit high, so $E = 0.1$ is more realistic). Notice that this simulation would also apply for a 1 Tesla field, except $E = 1$ would correspond to 10^4 V/m and the time step would proportionally shorter since the gyro-frequency would be 10 times higher. The $E \times B$ drift velocity would be the same in either case $v_{drift} = E/B = 0.1$ (ie 1/10 of the thermal speed) but the orbit would be tighter and the acceleration dv/dt would be higher. The proton velocity would be the same (a 1 eV proton has the same velocity regardless).

This is all worth checking with a test code. I think doing the problem in Mathematica is a straightforward first step and easy to plot orbits, change parameters, etc. As a check, you could turn off the electric field and you should just get circular Larmor orbits in a constant B field. If $v = B = 1$, then the orbit time should be unity (actually maybe 2π since $\omega = 2\pi/T$). The orbit radius should be unity also since $\rho_i = v/\omega_{ci}$. For a big run, we might consider using a compiled particle pushing code in more realistic, 3D SSX reconnection fields.

The goal will be to run a lot of particles in our 2D "Harris" configuration. Magnetic field varies like $tanh(x/\delta)\hat{y}$ and electric field varies like $sech^2(x/\delta)\hat{z}$. You might want to provide for a constant offset electric field so you can start particles off to the left and they'll have a little drift into the central electric field. The width of the reconnection electric field layer compared to the ion orbit will matter. Since the scaling for sizes is the proton gyro-radius, $\delta = 1$ corresponds to a width of a thermal proton orbit (less than 1 cm for 0.1 T and 1 eV). If the width is too narrow, the ions will zip through and not get much energy. You'll have to tinker but try $\delta = 1 - 5$. I think to start, it would be good to pick an initial constant energy (say 1 eV) and vary the initial phase. If the magnetic field is in the $\hat{y}$ direction, the orbits will be in the $x - z$ plane, so for $v = 1$ and $Z/M = 1$ (thermal proton) you might try 100 different initial directions. The energy after passing through the central electric field should be bigger than the initial. Once everything is working with protons, see if there's a difference with different Z/M. Try $Z/M = 1/1, 1/4, 2/12, 3/40$.