The Philosophical Quarterly, Vol. 53, No. 210 January 2003
ISSN 0031–8094

THE INDISPENSABILITY ARGUMENT AND MULTIPLE FOUNDATIONS FOR MATHEMATICS

By Alan Baker

One recent trend in the philosophy of mathematics has been to approach the central epistemological and metaphysical issues concerning mathematics from the perspective of the applications of mathematics to describing the world, especially within the context of empirical science. A second area of activity is where philosophy of mathematics intersects with foundational issues in mathematics, including debates over the choice of set-theoretic axioms, and over whether category theory, for example, may provide an alternative foundation for mathematics. My central claim is that these latter issues are of direct relevance to philosophical arguments connected to the applicability of mathematics. In particular, the possibility of there being distinct alternative foundations for mathematics blocks the standard argument from the indispensable role of mathematics in science to the existence of specific mathematical objects.

I. ASSESSING THE INDISPENSABILITY ARGUMENT

Ought we to include abstract mathematical objects in our ontology, and if so, which such objects should we include? Participants in this long-standing metaphysical debate have tended to fall into two reasonably well defined camps, nominalism and Platonism:

NOM. There do not exist any abstract (mathematical) objects

PLAT. There exist (at least) enough abstract objects to make literally true the bulk of the mathematical statements we accept.

The rhetoric on each side of the debate follows well worn paths. The nominalist argues that taking mathematics to be literally true commits us to a vast and mysterious realm of abstract entities. The existence of such a realm is at odds with what seems likely; hence denying the truth of mathematics is supported by common sense. The Platonist argues that denying the literal truth of mathematical claims is a radical sceptical move that offends against our firmly held beliefs that mathematics is a body of certain truths. Hence asserting the truth of mathematics is supported by common sense.

The nominalist characterizes the Platonist as a bizarre mystic, while the Platonist characterizes the nominalist as a crazy sceptic. Each side takes the burden of proof to lie with the opponent, resulting in deadlock.

Some philosophers have attempted to break this deadlock by considering the way in which mathematics is applied to the physical world. Central to this line of investigation has been the so-called indispensability argument. The basic idea behind the argument is straightforward. When faced with a general ontological question such as 'Do mathematical objects exist?', we should look to our best available theories of the world for guidance. Among our current best theories of the world are the theories of empirical science, and current science (especially physics) quantifies, seemingly unavoidably, over mathematical objects. Hence we have good reason to believe in the existence of mathematical objects, unless and until we can do science without postulating them. In summary,

1. We ought rationally to believe our best available scientific theories
2. Mathematics is indispensable for science
C. Therefore we ought to believe in the existence of (abstract) mathematical objects.

An interesting feature of the indispensability argument, and one reason, I think, why it has been taken so seriously by both nominalists and Platonists, is that it offers an approach to the debate over the existence of mathematical objects that does not (or at least does not obviously) beg the question against either side. On the one hand, the indispensability argument sides with nominalists in avoiding any presupposition that mathematical statements are intrinsically privileged. On the other hand, the argument sides with Platonists in taking mathematical statements at face value, as making genuine ontological claims (thus ruling out more radical anti-Platonist approaches such as formalism). This evenhandedness is an important strength of the indispensability argument, and I shall return to consider it in more detail towards the end of the paper.

I.1. *Refuting nominalism and establishing Platonism*

While the indispensability argument may be even-handed in its presuppositions concerning mathematics, its goal is clearly pro-Platonist. Its conclusion is that we ought to believe in the existence of abstract mathematical objects, and this rules out any purely nominalist ontology. Interestingly, however, criticism of the argument has come from both sides of the nominalist/Platonist debate. Nominalists have attacked each of the premises of the indispensability argument, hoping to demonstrate that the argument is unsound. Premise (1) is prescriptive in nature: it advocates realism and a certain kind of holism with respect to our best scientific theories. It may be resisted either by rejecting scientific realism, in the manner of van Fraassen, or by denying the holistic assumptions which place mathematical entities on an epistemological par with theoretical concrete entities in science.[1] Premise (2) makes an empirical claim, that mathematics is indispensable for science. Attempts at undermining this claim have tended to involve more detailed, technical work, using logical apparatus to construct mathematics-free formulations of portions of current science. This is exemplified by Hartry Field's extended project, outlined in his *Science Without Numbers*.[2]

The second kind of criticism of the indispensability argument has come from dissatisfied Platonists, worried that its conclusion falls short of full-blown Platonism. (PLAT) is not simply the contradictory of (NOM). Rather they are contraries: they cannot both be true, but they can both be false. Thus even if the indispensability argument succeeds in refuting nominalism, this does not thereby establish that Platonism is true. (For example, it might conceivably be the case that the indispensability argument entails the existence of only one mathematical object. This would refute nominalism, but leave the Platonist dissatisfied.) Platonist worries about the indispensability argument have largely centred on the strength and scope of its conclusion. Charles Parsons has criticized the argument for placing mathematics on an epistemological par with theoretical physics, and thus failing to explain the intuitive obviousness of mathematical truths.[3] This criticism can be seen as questioning the *strength* of the conclusions of the indispensability argument. Another claim which is often made by Platonist sympathizers dissatisfied with the indispensability argument is that it provides justification only for the scientifically applicable parts of mathematics. Even Quine, a staunch defender of the argument, has on occasion conceded that the higher reaches of set theory must according to the indispensabilist be viewed as a formal game.[4] The implicit criticism here is that the *scope* of the conclusions of the indispensability argument is insufficient to capture all of current mathematics.

I think there is considerable merit to these Platonist worries concerning the strength and scope of the conclusion of the indispensability argument. However, I am not convinced that these worries are sufficient to undermine

[1] The potential shortcomings of extending van Fraassen-style anti-realism to mathematics are discussed in O. Bueno, 'Empiricism, Mathematical Change and Scientific Change', *Studies in History and Philosophy of Science*, 31 (2000), pp. 269–96.

[2] H. Field, *Science Without Numbers: a Defence of Nominalism* (Oxford: Blackwell, 1980).

[3] C. Parsons, 'Quine on the Philosophy of Mathematics', repr. in his *Mathematics in Philosophy: Selected Essays* (Cornell UP, 1971), pp. 176–205.

[4] 'Reply to Parsons', in L. Hahn and P. Schilpp (eds), *The Philosophy of W.V.O. Quine* (La Salle: Open Court, 1986), pp. 396–403.

the *prima facie* support which the indispensability argument provides for Platonism. (PLAT) claims that there exist enough abstract objects to make literally true the bulk of the mathematical statements we accept. This is an ontological claim about the existence of abstract objects with certain features; the modal and epistemological strength of these claims is a separate issue. This claim may still remain intact even if we resist classifying as *a priori* the resulting knowledge we have of these objects, which is one way of responding to the assertion that the indispensability argument places the epistemological strength of mathematical claims on a par with the theoretical claims of empirical science. Concerning the scope objection, it is easy to forget how much of mathematics is applied in science in one form or another. Moreover, there are numerous examples of seemingly useless mathematical theories finding subsequent application. The applications of group theory to particle physics and of quaternions to quantum mechanics provide two striking examples. This suggests that the truly inapplicable portions of mathematics will be so peripheral that the central thesis of (PLAT) may plausibly be maintained.[5]

My own view is that Platonists are right to be dissatisfied with the indispensability argument, but that they are wrong over where they have focused this dissatisfaction. The real problem with the indispensability argument lies not in the strength or scope of its conclusions but in their *specificity*. I shall argue that the argument almost certainly cannot be used to generate specific ontological conclusions. The problem stems from the likelihood of there being distinct alternative ontological foundations for mathematics. I shall show that this possibility blocks any straightforward move from the indispensability of mathematical structures to the indispensability of particular mathematical objects. In that case the indispensability argument cannot be used to establish (PLAT).

I.2. *Indispensability and ontology*

(PLAT) is at root an ontological thesis. Thus if the indispensability argument is to establish (PLAT) convincingly, the ontological ramifications of the argument must be made explicit. The key is the indispensability claim

2. Mathematics is indispensable for science.

The roots of the indispensability argument lie in Quine's notion of ontological commitment, and Quine originally defined this notion not in terms of indispensability but in terms of quantification over objects. Using this terminology, (2) becomes

[5] For further discussion of this issue, see M. Colyvan, *The Indispensability of Mathematics* (Oxford UP, 2001), p. 107.

2*. All alternative formulations of our best available scientific theories quantify over (abstract) mathematical objects.

Let T range over alternative formulations of our optimal scientific theories, and M range over collections of mathematical objects. Then (2*) can be logically paraphrased as follows:

2*. $\forall T \exists M$(T quantifies over M).

This implies that we cannot formulate current science without quantifying over *some* mathematical objects, but it does not imply that there is any particular collection of mathematical objects which is required for current science. Thus in the context of the argument as a whole, (2*) suffices to refute (NOM), but is insufficient to establish (PLAT). The reason why not is that establishing (PLAT) requires establishing the existence of a particular, and specified, collection of mathematical objects. What the Platonist needs to establish, therefore, is

P*. $\exists M \forall T$(T quantifies over M)

where the order of quantifiers is reversed, and where M is a collection of mathematical objects extensive enough to underpin our central mathematical claims.

I should stress that I am not intending to downplay (2*). If the Platonist can establish (2*), then that would be a significant result, not least because it knocks (NOM) out of the running. Thus if the indispensability argument can be used to derive (2*), and even if nothing stronger or more specific can be derived, this would still be of considerable philosophical importance. Nevertheless, although (2*) tells us that some mathematical objects exist, it does not tell us which ones. And a fully developed version of (PLAT) ought presumably to be able to tell us precisely what mathematical objects exist and provide a fairly detailed story about their nature. The question becomes, therefore, whether some premise along the lines of (P*) can be established. In other words, is there some collection of mathematical objects over which all adequate alternative formulations of our best scientific theories quantify? I shall argue in §II that the answer to this question is 'No', for reasons connected to the multiple reducibility of mathematical objects.

I.3. *Structure and the application of mathematics to science*

The argument I shall be presenting depends in part on the following preliminary claim, that what are indispensable (or dispensable) for science in the first instance are not mathematical objects but mathematical *structures*. When a scientific theory quantifies over real numbers, for example, it is exploiting the structure of the real number line in order to make assertions and predictions about the physical world. Unlike concrete objects, individual mathematical objects play no theoretical role independently of the structures in which they are embedded. This is partly a consequence of the abstract, and hence presumably acausal, nature of mathematical objects. This is in sharp contrast to the theoretical role of concrete objects. A single astronomical object, say, a black hole, may be postulated to explain a particular set of observations, without necessarily being linked to any other such objects. Yet it would be strange to imagine a particular mathematical object, say, the number 17, being quantified over by a scientific theory without the rest of the mathematical objects which complete the mathematical structure of which 17 is a part also playing some theoretical role.

This is a familiar enough point, and I shall not labour it. However, it is important to distinguish the above position from the cluster of related philosophical views that fall under the heading of mathematical structuralism. My point is not that we need to agree that mathematical theories are ultimately *about* structures, but rather that it is solely the structural features of mathematical theories that are relevant to their use in science. This is quite consistent with the traditional Platonist position encapsulated in (PLAT), according to which mathematical theories carry with them certain ontological commitments to mathematical objects.

As discussed in the previous section, the indispensability argument will only yield a specific ontological conclusion if it can be established that there are particular mathematical objects which must be quantified over by any adequate formulation of our best scientific theories. The *quasi*-structuralist perspective I am proposing allows investigation of this issue to be broken down into two component tasks. The first task is to ascertain the structural mathematical requirements of science, essentially an empirical question. The second task is to determine what mathematical objects are required to underpin these structures, which is a logico-mathematical question.

To take the empirical question first, the structural requirements especially of advanced theories in physics, such as quantum mechanics and general relativity, appear considerable, invoking as they do such varied and complex structures as infinite-dimensional vector spaces, tensors, quaternions and Fourier series. Since my ultimate concern is to establish the in-principle limitations of the indispensability argument as a pro-(PLAT) argument, I shall assume for sake of argument that the empirical question has been settled in the Platonist's favour. I shall assume, in other words, that science has significant structural mathematical requirements, including at least as much structure as is necessary for real analysis.

This leaves the second, logico-mathematical question. To what specific mathematical objects is science ontologically committed in virtue of these structural requirements? In what follows I shall argue that the answer to this question is 'None'. My argument hinges on the phenomenon of *multiple reducibility*. This concept is not a new one: indeed, it has attracted considerable discussion among philosophers interested in the metaphysics of mathematical objects. However, the form of multiple reducibility that is most relevant to indispensability has been largely ignored. It is the possibility that mathematics, or at least its applied portions, is reducible to multiple and distinct ontological foundations that poses the most serious threat to extracting specific ontological conclusions from the indispensability argument.

II. MULTIPLE REDUCIBILITY

II.1. *The standard multiple reducibility problem*

Interest in the metaphysical difficulties posed by the phenomenon of multiple reducibility in mathematics was sparked by Paul Benacerraf's 1965 article 'What Numbers Could Not Be'.[6] Benacerraf focuses his attention on natural numbers, and he begins with the observation that there are multiple different (and conflicting) ways of 'reducing' numbers to sets. For example, the natural numbers can be identified with either of the following two ω-sequences of sets:

(a) $\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, \ldots$
(b) $\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}, \ldots$

together with an appropriate definition of the successor relation in each case. On the first reduction, the number 2 is identified with the set $\{\{\varnothing\}\}$; on the second reduction, the number 2 is identified with the set $\{\varnothing, \{\varnothing\}\}$. If the number 2 is a genuine object, then it cannot be identical with both of these two distinct sets. Yet there seems to be nothing to choose between these two reductions – in each case Peano arithmetic is perfectly embedded in set theory.

What are the ramifications for the indispensability argument of the Benacerrafian notion of multiple reducibility? I shall argue that the indispensability argument is left more or less untouched. The issue I am addressing is whether specific ontological conclusions can be derived from the indispensability argument. To consider first the ontological status of sets,

[6] P. Benacerraf, 'What Numbers Could Not Be', *Philosophical Review*, 74 (1965), pp. 47–73.

the fact that Peano arithmetic is reducible to set theory means that any mathematical task that is performed by natural numbers can be adequately performed by a suitably chosen sequence of sets. Hence sets can act as a basic ontological foundation for Peano arithmetic. The fact that numbers are reducible to sets in *multiple* ways does not alter the basic situation. All that changes is that instead of there being just one adequate alternative formulation of current science, there are multiple adequate alternatives which differ only in which ω-sequence of sets they use as surrogates for the natural numbers. It may still be the case that set theory forms a basic and indispensable foundation for applied mathematics.

What about the ontological status of numbers? Here we enter murkier territory. The issue of what metaphysical conclusions ought to be drawn from the reducibility of one kind of object to another kind is a wide-ranging and contentious topic. One view is that the reducibility of Peano arithmetic to set theory means that quantification over numbers is not indispensable for science. We never need to quantify over numbers to do science, because we can always quantify over sets instead. Hence science is not ontologically committed to numbers, and there is no reason based on the indispensability argument to believe in the existence of numbers. However, there is room to dispute this conclusion. Some may claim that we are continuing to quantify over numbers after such a reduction, albeit in set-theoretic guise. Have we eliminated numbers or explained more about them? My point is that the fact that numbers are reducible to sets in *multiple* ways does not alter these basic issues concerning the relationship between reduction and indispensability. Deriving from the indispensability argument an ontological commitment to numbers, especially to numbers considered as *sui generis* mathematical objects, is problematic anyway, since the relation between reduction and indispensability is unclear.

Mark Colyvan, a staunch advocate of the Quine–Putnam indispensability argument, claims (p. 142) that the indispensability argument leaves open the issue of the precise metaphysical nature of mathematical objects, including whether numbers and sets are *sui generis* or are constituted by items such as universals or structures. While I agree with Colyvan's general point concerning the metaphysical agnosticism of the indispensability argument, its applicability to numbers in particular only gets off the ground if the existence of numbers is indeed implied by the argument. Otherwise the point that the indispensability argument is consistent with numbers being *sui generis* is moot. Colyvan also argues that Quine's belief in the existence of sets and not numbers is due to his 'extreme Ockhamist tendencies', rather than to the dictates of the indispensability argument. It is not clear to me that Ockhamism and the indispensability argument can be so neatly disentangled. In particular, the normative claim that we should ontologically commit ourselves to all and only those entities that are indispensable for our best theories bears more than a passing resemblance to the Ockhamist injunction to avoid multiplying entities beyond necessity.

The diagram below (Fig. 1) illustrates the basic structure of the multiple reduction scenario discussed by Benacerraf. *T* is Peano arithmetic, *S* is Zermelo–Fraenkel set theory, and the arrows represent distinct ways of embedding arithmetic in set theory.

multiple reducibility:
the standard case

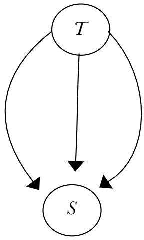

Figure 1

To summarize, the multiple reduction of Peano arithmetic to set theory may be enough to undermine an indispensability argument for the existence of numbers (depending on one's view of the metaphysics of reduction). However, it does not block an indispensability argument for the existence of *sets*. Moreover, any serious case for indispensability is likely to focus on sets rather than numbers. And the indispensability or otherwise of sets is seemingly unaffected by Benacerraf's multiple reducibility argument.

II.2. *Multiple reducibility as a threat to the indispensability argument*

The real threat to the specificity of the indispensability argument is not the multiple reducibility of Peano arithmetic to set theory, but the fact that there are *alternatives* to ZF set theory available which can also act as foundations for arithmetic. It is this potential lack of *unique* foundations for mathematics which threatens to block any attempt to make the conclusions of the indispensability argument more specific. ZF set theory is non-unique in this respect because there are alternative theories, not based on sets, which may also act as foundations for arithmetic and other areas of applied mathematics.[7] One example is category theory. A number of mathematicians and logicians have argued that category theory, or the category of categories, provides a foundation for mathematics. This view has not gone unchallenged. In a recent paper, Landry distinguishes at least two alternative positions concerning the status of category theory.[8] One view, attributed to John Mayberry, is that category theory cannot act as a foundation for mathematics, because it too requires set-theoretic notions to underpin it. A second view, attributed to Saunders Mac Lane, is that category theory plays

[7] Another source of potential foundations are theories based on other concepts of set, for example non-well-founded sets. §IV includes a brief discussion of this issue.

[8] E. Landry, 'Category Theory: the Language of Mathematics', *Philosophy of Science*, 66, Supp. 3 (2000), pp. S14–27.

an organizational role rather than a foundational role, picking out the common structural elements in different branches of mathematics.[9]

For the remainder of the paper I shall simply adopt it as a working hypothesis that there are multiple ontological foundations for mathematics. I shall use the particular examples of set theory and category theory to illustrate my points, but nothing I say will depend on the technical details of either of these theories. Diagrammatically, the situation is as in Fig. 2. The T_i are mathematical theories which are, by assumption, (structurally) indispensable for science, S is Zermelo–Fraenkel set theory, and C is category theory.

Is set theory indispensable for science, given the multiple foundation (MF) hypothesis? Category theory is not an extension of set theory, nor *vice versa*, and the ontologies of the two theories are entirely non-overlapping. Thus neither set theory nor category theory is indispensable for science, because neither provides a unique foundation for the scientifically applied parts of mathematics. Hence we are not rationally compelled to believe in the existence of sets, nor are we rationally compelled to believe in the existence of categories. Our ontological commitment to mathematical objects cannot be made more specific than a disjunctive commitment to sets-or-categories. If, as seems likely, each of category theory and set theory can serve as adequate and independent foundations for applied mathematics, the ramifications for (PLAT) are immediate and discouraging. Commitment to a specific ontology of abstract mathematical objects cannot be derived from the indispensability argument alone.

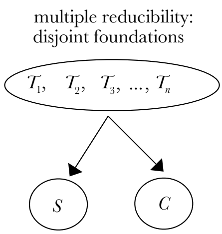

Figure 2

III. EXTENDING THE INDISPENSABILITY ARGUMENT

III.1. *Quine and the Platonist predicament*

The consequences of the MF hypothesis for the predicament of a Platonist who is relying on the indispensability argument can be summarized schematically as follows. Let A be a theory which is indispensable for science, and let B and C be two distinct theories to which A can be reduced. (For example, let A be Peano arithmetic, B be ZF set theory, and C be category theory.) The indispensability argument can be used to justify literal belief in the ontology of A, but the Platonist wishes to go beyond this and

[9] For a useful introduction to the mathematical applications of category theory, see C. McLarty, *Elementary Categories, Elementary Toposes* (Oxford: Clarendon Press, 1995).

argue specifically for the ontology of *B*. My claim is that there is no legitimate way to do this.

The writings of W.V.O. Quine present a *prima facie* challenge to my position. Quine started out as a committed nominalist, until a growing realization of the apparent indispensability of mathematics across a range of logical, philosophical and scientific contexts converted him to Platonism. Furthermore, he uses indispensability considerations to argue for a very specific ontological conclusion, namely, that we ought to believe in classes (and only classes, as far as *abstracta* are concerned). How, then, does Quine argue from the indispensability of mathematics to the existence of classes? A perusal of his scattered remarks on this issue indicates at least three distinct lines of argument. It will be instructive to examine each of these in order to see whether they hold up under the assumption of the reducibility of mathematics to multiple distinct foundations.[10]

Quine's first strategy is to argue that classes are themselves indispensable for science:

> Ways of serving the theoretical purposes of infinitesimals and ideal objects were found which did not call for these troublesome objects after all, and the objects were accordingly dropped. On the other hand no similar circumvention of classes suggests itself; one is impelled rather to the opposite course, that of keeping classes and coping with the trouble they make.[11]

The key claim of this first strategy is directly contradicted by the MF hypothesis. According to this hypothesis, category theory is an adequate foundation for applied mathematics; hence a 'circumvention' of classes is indeed available.

Quine also presents a second line of argument for the existence of classes which is based on the *sufficiency* of classes as a foundation for the rest of mathematics.

> The classes thus posited are, indeed, all the universals that mathematics needs. Numbers, as Frege showed, are definable as certain classes of classes. Relations, as noted, are likewise definable as certain classes of classes. And functions, as Peano emphasized, are relations.[12]

According to the MF hypothesis, each of set theory and category theory provides enough objects of the appropriate kind to underpin all of the

[10] A terminological note: Quine's use of the term 'class' refers to the objects postulated by the version of set theory which he favours. For the purposes of this discussion, 'class' and 'set' can be regarded as synonymous. The multiple reducibility assumption can therefore be understood as the assumption that both classes and categories are adequate to underpin all of applied mathematics.

[11] Quine, *Word and Object* (MIT Press, 1960), p. 266.

[12] Quine, *From a Logical Point of View* (Harvard UP, 1953), p. 122.

mathematics applied in science; so, in the above terminology, each theory is *sufficient* to underpin the whole of applied mathematics. Thus the sufficiency claim is factually correct; but why should sufficiency imply ontological commitment? If there were a multitude of distinct ontological foundations for mathematics, should we adopt the ontologies of all of them? Quine's sufficiency argument is a licence for potentially unrestrained ontological inflation. It therefore seems strange that Quine, as an enthusiastic advocate of Ockham's razor, should want to endorse it.

III.2. *Pragmatic considerations*

To recapitulate, Quine's arguments from necessity and from sufficiency do not overcome the barrier which the MF hypothesis raises to establishing (PLAT). This outcome seems inevitable, given the structure of the problem. *A* is a theory which is indispensable for science, and *B* and *C* are two distinct theories to which *A* can be reduced. Extending the indispensability argument is fruitless, since *B* is not strictly indispensable for *A*. Arguing for *B*'s ontology on the grounds that the full theory *A* is reducible to *B*, and hence that *B* is sufficient for *A*, does not work either, since an exactly parallel argument can be used to justify belief in the distinct ontological commitments of *C* as well.

Quine's third and most interesting strategy for defending the inclusion of classes in our ontology is to extol the practical or 'pragmatic' benefits which they bring to our theories. The alleged pragmatic benefits of postulating classes are many and varied, but three thematic benefits to which Quine repeatedly alludes are power, fruitfulness and simplicity. Allowing ourselves the apparatus of classes and quantification over classes, we can construct mathematical and scientific theories which are more powerful, simpler and more fruitful than those which do not feature classes. This, Quine believes, is enough to show that we ought rationally to believe in the existence of classes.

In the schematic terms introduced at the beginning of §III, Quine's pragmatic strategy has the following form. We are looking for a way to break the deadlock between *B* and *C*. If *B* turns out to be superior to *C* on pragmatic grounds, then Quine has an argument for belief in *B*, as follows: *A* is indispensable for science, hence we ought to believe in the literal truth of *A*. *A* is reducible both to *B* and to *C*, each of which offers pragmatic advantages over *A*. But theory *B* is pragmatically superior to *C*, and hence we should prefer the reduction of *A* to *B*. Hence we have reason to believe in the existence of the objects quantified over by *B*.

Is this strategy sufficient to establish commitment to a specific ontology of classes? The first point to make is that Quine's pragmatic strategy is essentially a more nuanced version of the necessity strategy I considered previously. The underlying thesis of the necessity argument was that classes are indispensable for science. Given the MF hypothesis, this thesis becomes untenable. The underlying thesis of the pragmatic argument is that classes offer the pragmatically best mathematical basis for science, even if they do not offer the only adequate basis. The issue, then, is whether this weaker claim can be sustained in the face of multiple reducibility.

As before, I shall assume that the empirical situation is as favourable as possible for the Platonist. In other words, I shall assume that scientific theories based on classes *are* pragmatically superior to any purely nominalist reconstructions of science. This leaves the following question to be answered: are the pragmatic considerations which Quine wields successfully against the nominalist sufficient also to single out a unique (and specific) best foundation for mathematics among the mathematical alternatives? I can sharpen this question by considering a specific situation involving some applied mathematical theory, for example, real analysis. Does the choice whether to reduce real analysis to set theory or to category have any effect on the pragmatic effects this theory has on science?

The sorts of pragmatic benefits which Quine is interested in are power, fruitfulness and simplicity. It is certainly possible to see how benefits of this sort might flow from foundational work in mathematics. One aim of such work is to provide a rigorous basis for specific mathematical theories. Establishing the rigour of a mathematical theory may in turn increase its power as a scientific tool by ensuring that use of the theory for scientific applications will not lead to logical error. Another aim of foundational work is to provide a common basis for hitherto independent mathematical theories. This may allow techniques established in one theory to be transferred to another theory, thereby increasing the fruitfulness of applying mathematics to scientific ends.

The above considerations lend support to Quine's pragmatic strategy. However, there is a *caveat* over two important points. First, it is one thing to outline potential pragmatic benefits for science arising out of foundational work in mathematics; it is quite another to come up with actual examples from scientific practice.[13] Secondly, even if such examples can be found, all they would show is that a mathematics with proper foundations has pragmatic benefits for science as a whole, compared to a mathematics

[13] One potential historical example is the rigorous foundations for the infinitesimal calculus provided in the early nineteenth century by Cauchy, Weierstrass, and others. Prior to this rigorization, the use of infinitesimals in mechanical and dynamical applications was plagued by potential pitfalls. However, it is less clear that the nineteenth-century developments had much effect on the practical use of the calculus in science.

without any such foundations. What I am interested in is whether these pragmatic benefits differ (or might plausibly differ) between alternative choices of mathematical foundation. My working hypothesis, to which I have been referring as the MF hypothesis, is that each of set theory and category theory provide fully adequate foundations for applied mathematics. Presumably part of their being adequate is that each meets accepted standards of rigour. Moreover, each is a single overarching theory, and will thereby provide a common basis for every individual applied mathematical theory. Hence the potential pragmatic benefits discussed in the previous paragraph seem to be common to both foundational theories – set theory and category theory – given the backdrop of the multiple reducibility assumption. My worry therefore is not that providing foundations for mathematics will have no pragmatic consequences for science, but that any such consequences may be effectively the same for all adequate alternative foundations. If so, then Quine's pragmatic strategy will not succeed in overcoming the multiple reducibility problem and establishing a specific ontological foundation for (PLAT).

III.3. *Strong mathematical naturalism*

One response to the above objection is to concede that the choice between alternative adequate mathematical foundations has no external repercussions for science, but to argue that there are distinct pragmatic consequences *within* mathematics, and that these are enough, first, to justify a particular choice of foundation, and secondly, to allow ontological conclusions to be drawn. The first half of this claim is quite plausible. For example, set-theorists debate which, if any, independent axioms should be added to Zermelo–Fraenkel set theory. The axiom of choice and the continuum hypothesis may potentially have ramifications for scientific applications, but as the debate moves into the realm of large cardinal axioms, it seems increasingly likely that any choice between them will only impinge on pure mathematics. Similarly, a systematization of mathematics based on category theory may well have different pragmatic consequences within mathematics to a systematization based on set theory.[14]

I shall assume, then, that deferring to internal mathematical considerations succeeds in breaking the tie between set theory and category theory. Say they favour set theory as an ontological foundation for applied

[14] See, for example, C. McLarty, 'The Uses and Abuses of Topos Theory', *British Journal for the Philosophy of Science*, 41 (1990), pp. 351–75, at p. 370: 'If categorical foundations, or any other foundations, are accepted then we might benefit from the very fact that one rigorous conception has made way for another. ... like set theory in its time, category theory arose from the heart of mathematical practice and offered foundational insights.'

mathematics.[15] What about the second half of the above claim, namely, that this is enough to justify belief in the existence of sets? The presumption is that it is legitimate to defer to the judgements of mathematicians, made on the basis of factors internal to mathematics, in order to justify drawing ontological conclusions from a particular choice of foundational theory. One of the underlying assumptions of the basic indispensability argument is a certain sort of scientific naturalism. Although this stance is by no means universally shared by philosophers in general, both Platonists and nominalists may in principle agree that our best scientific theories (as judged by scientists) carry ontological weight, without thereby contradicting their respective positions. The corresponding stance with respect to mathematics is sometimes referred to as *mathematical naturalism*.

At this juncture, a brief terminological excursion is necessary. The most prominent contemporary exponent of mathematical naturalism is Penelope Maddy.[16] Maddy advocates deferring to mathematicians concerning issues that are internal to mathematics, including foundational issues such as choosing whether to adopt certain set-theoretic axioms. However, she resists drawing ontological conclusions from the judgements of the mathematical community. Her reasons for not doing so stem from her assertion that ontological and metaphysical questions concerning mathematics, including the debate between (NOM) and (PLAT), 'float free' from actual mathematical practice, and that the answers to such questions neither determine nor are determined by the activities and judgements of mathematicians. My purpose here is neither to attack nor defend Maddy's brand of mathematical naturalism, but to distinguish it from the version in which I am interested at present. What the internal/external distinction shows is that there are at least two ways of mirroring scientific naturalism in the domain of mathematics. Both versions agree that there is no privileged, external 'tribunal' for settling mathematical questions. Thus both take the collective pronouncements of mathematicians seriously. On Maddy's version, 'taking seriously' involves drawing ontological conclusions only if those conclusions are themselves part and parcel of mathematical practice. On the alternative version, 'taking seriously' amounts to 'taking at face value'. So if mathematicians agree that there are non-denumerable sets, then we should include non-denumerable sets in our catalogue of what there is. I shall call this

[15] It is perhaps debatable whether Quine would accept the legitimacy of the concept of 'internal mathematical reasons', given his repudiation of any sharp dividing line between empirical science and mathematics. However, in his response to Parsons (in Hahn and Schilpp, p. 399), Quine agrees that purely mathematical statements can be characterized as statements in which only mathematical vocabulary occurs essentially.

[16] The fullest exposition of Maddy's view is to be found in her book *Naturalism in Mathematics* (Oxford: Clarendon Press, 1997).

second version *strong mathematical naturalism*, to distinguish it from Maddy's weaker, non-ontological version.[17] It is strong mathematical naturalism which is being appealed to in the above amended version of Quine's pragmatic strategy, in order to justify belief specifically in an ontology of sets.

This brings me to the core of my objection against this modified pragmatic strategy. My concern is not that the strategy's presupposition of strong mathematical naturalism is incoherent, or impossible to justify adequately as a stance. The problem is that strong mathematical naturalism cannot be grafted onto the indispensability argument without giving up a key strategic advantage of the argument as a whole. This advantage of the indispensability argument is that it offers to the debate between (PLAT) and (NOM) an approach based on presuppositions which are acceptable, at least potentially, to both sides. By contrast, strong mathematical naturalism is manifestly incompatible with (NOM).

Perhaps there are good arguments for strong mathematical naturalism, maybe even arguments persuasive enough to induce nominalists to abandon their ontological viewpoint. In that case my concern becomes a different one. If compelling reasons can be found to support strong mathematical naturalism, why can we not appeal to these same reasons at the outset of our ontological enquiry, and thereby bypass the indispensability argument altogether? In other words, if the indispensability argument is *sufficient* to establish full-blown Platonism, then it is not *necessary* for this task. The only way to extract a commitment to specific mathematical objects from the indispensability argument is to appeal to internal mathematical considerations in order to break the tie between alternative ontological foundations. However, if these considerations carry ontological weight, then we could simply have appealed to them directly, without taking a detour through the indispensability debate, in order to support the inclusion of mathematical objects in our ontological catalogue.

IV. CONCLUSIONS

I have argued that the likelihood of there being multiple adequate foundations for mathematics presents an insuperable obstacle to using the indispensability argument to establish full-blown Platonism, as in (PLAT). The reason is that in this case, even if mathematics is indispensable for science, no particular collection of mathematical objects is indispensable.

[17] In an earlier paper, 'Indispensability and Practice', *Journal of Philosophy*, 89 (1992), pp. 275–89, esp. p. 280, Maddy tentatively endorses a position very much like the strong mathematical naturalism that I am discussing.

For example, it may be possible to reduce mathematics to set theory, or to reduce it to category theory. It is also difficult to see how a choice between alternative foundations for mathematics, assuming that each is adequate, can have significant ramifications outside mathematics. Hence if a principled choice between the alternative foundations is to be made, it can only be made on internal mathematical grounds. To draw ontological conclusions on these grounds is to adopt a version of mathematical naturalism which I have labelled 'strong mathematical naturalism'. There are two possibilities here. Either strong mathematical naturalism cannot be justified, in which case the indispensability argument is insufficient to establish (PLAT). Or strong mathematical naturalism can be justified, in which case we can simply infer the truth of (PLAT) directly from it without the need for the indispensability argument.

Without doubt, if the indispensability argument is sound then its conclusion that abstract mathematical objects exist is of great significance. It would show that nominalism, (NOM), is false. Yet a Platonism that cannot say anything about which specific mathematical objects exist is unsatisfying.

The distinction between mathematical objects and mathematical structures is an important one. Part of my argument is that the likelihood of there being multiple foundations for mathematics implies that it is primarily mathematical structures rather than mathematical objects which are indispensable for science. This conclusion fits well with mathematical structuralism, though it does not entail such a position. Stewart Shapiro distinguishes two fundamental versions of mathematical structuralism.[18] According to '*in re* structuralism', structures are ontologically dependent on the systems which exemplify them. According to '*ante rem* structuralism', structures are prior to and independent of any specific systems or collections of objects. Combining either version of mathematical structuralism with the indispensability argument yields the conclusion that the structure of the natural numbers exists. The complications arising from the multiple reducibility of numbers to either sets or categories can therefore be bypassed. However, the problem I have raised, concerning deriving a specific foundational commitment to either sets or categories from the indispensability argument, remains even when we shift from Platonism to structuralism. For neither the structure of ZF set theory nor the structure of category theory is indispensable for science.

I conclude with three disclaimers. First, as I pointed out in my original discussion of the indispensability argument, there has been some recognition in the literature of potential shortcomings of the indispensability argument

[18] S. Shapiro, *Philosophy of Mathematics: Structure and Ontology* (Oxford UP, 1997), pp. 84–90.

for establishing full-blown Platonism. However, I believe that by focusing on the strength and scope of the indispensability argument, previous criticisms of the argument have missed what is potentially its most serious weakness, namely, its specificity. The linking of indispensability to the issue of multiple foundations for mathematics is not an area that has been previously explored in any detail. Even Quine, who in his later writings increasingly stressed the notion of ontological relativity, seems to have underestimated the challenges of establishing a specific mathematical ontology on the basis of indispensability considerations. It might be objected that the fact that Quine appeals to pragmatic considerations in justifying a particular foundational choice indicates that he realizes that the indispensability argument cannot do the ontological work unaided. But for Quine, pragmatic reasons are as good reasons as any. Moreover, I have argued that even if the indispensability argument is supplemented in this way, it cannot overcome the barrier of multiple alternative foundations and establish full-blown Platonism.

My second disclaimer is to reiterate that my argument is based on a hypothetical assumption, namely, that there are multiple distinct foundations for applied mathematics. If this assumption turns out to be false, then (PLAT) may be salvageable using the indispensability argument. Nevertheless I take it that the assumption, if not conclusively verified, is at least highly plausible. I have used set theory and category theory as my examples of potential alternative foundations, because these are alternatives which have been seriously proposed. The debate in particular over the status of category theory, and whether its foundational role can be truly separated from set theory, is ongoing.[19] However, my point is a more general one, and is not dependent on the outcome of any particular foundational debate. Indeed the point can be made, albeit less dramatically, even within the ontological framework of set theory. There has been, and continues to be, debate among logicians and set theorists concerning the addition of independent axioms to ZF set theory, ranging from the continuum hypothesis and the axiom of choice up to large cardinal axioms. Although it is dangerous to make sweeping open-ended claims, it seems likely that in many cases the choice between adopting one large cardinal axiom or another, for example, will have no ramifications, pragmatic or otherwise, outside pure mathematics.

My third and final disclaimer concerns mathematical naturalism. I have argued that without the assumption of strong mathematical naturalism, the indispensability argument is insufficient to establish Platonism, and with this

[19] For further useful references concerning this issue, see the bibliographies in McLarty, *Elementary Categories*, and Landry, 'Category Theory'.

assumption it is unnecessary for establishing Platonism. However, there is one other possibility. There might be some argument from the indispensability of mathematics to the acceptability of strong mathematical naturalism. If so, then this newly established mathematical naturalism could be used to sharpen the ontological conclusions of the indispensability argument. Having said this, I fail to see how any convincing argument of this sort might proceed. Why does establishing that *some* mathematical ontology is required for science thereby give ontological legitimacy to methodological and pragmatic considerations internal to mathematics and to mathematicians?[20]

Xavier University, Ohio

[20] Thanks are due to Paul Benacerraf, Gideon Rosen and Mark Colyvan for helpful comments on earlier drafts of this paper.