Science-Driven Mathematical Explanation

ALAN BAKER

Swarthmore College

1. Introduction

1.1 Mathematical explanation

The vast majority of philosophical work on explanation has concerned itself with scientific explanation. Aside from the obvious importance of science, another factor sometimes cited in support of this partiality is that there is 'a substantial continuity between the sorts of explanations found in science and at least some forms of explanation found in more ordinary non-scientific contexts' (Woodward 2009, p. 2). The idea seems to be that, by focusing on explanation in science, philosophers will be able to isolate and analyze features of explanatory practice that hold more generally.

A notable exception to the above claim of continuity is explanation in mathematics. This topic was entirely ignored in the 'first wave' of work by analytic philosophers on explanation in the 1950's and 1960's. The situation has changed considerably over the past couple of decades, and there is now a significant amount of philosophical attention being paid specifically to the issue of mathematical explanation. Moreover, even philosophers whose main focus is on scientific explanation often acknowledge the existence of explanations within pure mathematics. Nonetheless, the predominant view is that mathematical explanation is qualitatively different both from scientific explanation and from explanation in 'ordinary non-scientific contexts'.[1] Thus the editors of a recent volume on interdisciplinary approaches to explanation write that 'your explanation of why you'll be home late for dinner and a mathematician's proof of a theorem share very little' (Wilson & Keil 2000, p. 91).

There are good reasons for analyzing mathematical explanation separately from scientific explanation. Adopting this approach acknowledges the clear intuitive differences between these two spheres of explanatory practice, while also allowing theorists in one sphere to proceed unencumbered by potential counterexamples from the other. However, there is a fairly obvious problem with this neat picture, and that is that the spheres of mathematical practice and scientific practice frequently *overlap*. It is all very well emphasizing the qualitative differences between scientific explanation and its mathematical counterpart, but what about scientific explanations that make use of mathematics?

Following Paolo Mancosu, I shall define a mathematical explanation in science (MES) to be 'an explanation in natural science carried out by essential appeal to mathematical facts' (Mancosu 2008, p. 135). An example of an MES that has been discussed at some length in the literature concerns the life-cycle of the North American

[1] One line of dissent from this view are unification-based accounts of explanation, as developed by Kitcher and others. See Kitcher (1989).

forthcoming in *Mind*

periodical cicada.[2] Cicadas are locust-like insects which spends many years underground in larval form before briefly emerging as an adult. There are two subspecies of periodical cicada, one with a 13-year life-cycle and the other with a 17-year life-cycle. Biologists wondered if there was some reason why both subspecies have life-cycle period lengths that are prime. Surprisingly, it turns out that there is a candidate explanation for this phenomenon that is based on number theory. The idea is that by adopting a prime life-cycle period, cicadas minimize the frequency with which they intersect with periodical predators. For example, a (hypothetical) 10-year cicada would coincide on each emergence with 2-year, 5-year and 10-year predators, whereas an 11-year cicada would only coincide with 11-year predators. The number-theoretic basis for the explanation is the fact that the lowest common multiple of two numbers is maximal when the numbers are co-prime.

Another example of an MES, discussed by Mark Colyvan, is from the domain of meteorology:

> We discover at some time t_0 there are two antipodal points p_1 and p_2 on the earth's surface with exactly the same temperature and barometric pressure. What is the explanation for this coincidence? (Colyvan 2001, p. 49)

The explanation for this sameness lies in a corollary of the Borsuk-Ulam theorem from algebraic topology, which implies that whenever there are two smoothly varying quantities defined on the surface of a sphere there exists at least one pair of antipodal points for which the values are equal for each of the two quantities.

1.2 Analysing mathematical explanation in science

The topic of mathematical explanation in science has been the focus of considerable attention in the recent philosophical literature.[3] This attention has come almost exclusively from philosophers of mathematics, in particular those interested in the platonist / nominalist debate concerning the nature and existence of mathematical objects. Mathematical platonists such as Mark Colyvan have sought to refine Quine's 'Indispensability Argument' for the existence of (abstract) mathematical objects, based on the indispensability of mathematics in empirical science, by focusing on the potential *explanatory* role of mathematics in science. There is growing consensus to be found in this literature that there are indeed genuine mathematical explanations of physical phenomena, although what this implies for the opposing positions in the ontological stand-off is very much in dispute. Issues of contention include the precise role of inference to the best explanation as a methodological principle in science, whether this

[2] The example was first presented and discussed in Baker (2005). See also Leng (2005), Bangu (2008), Mancosu (2008a, 2008b), Daly & Langford (2009), Baker (2009), and Saatsi (forthcoming).

[3] The beginning of the current wave of philosophical interest in MES can be traced back to Colyvan (2002) and Melia (2002), as well as to books on related topics by Batterman (2002) and Morrison (2000). See Baker (2005) for an overview of the debate.

cuts across the concrete / abstract divide, and whether acknowledged fictional entities can play a genuine explanatory role.

A striking feature of this recent literature is the almost total absence of any analysis of just what kind of explanatory relation is involved in a typical MES. This is a puzzling lacuna since the availability of an analysis of this sort would enable the discussion to move beyond the trading of intuitions about individual candidate cases of MES. Nor is the lack of analysis of MES a problem just for the ontological debate within the philosophy of mathematics. It has also contributed – as I hope to show – to the tendency of philosophers working on general accounts of scientific explanation to set aside mathematical explanation (including MES) and bracket it off as 'not their problem'. There are some recent signs that this attitude is changing. For example, Colyvan (2010) ends with the following declaration: '[t]he debate over platonism and nominalism would be genuinely advanced by a better understanding of explanation – especially those explanations that have mathematics playing the leading role' (Colyvan 2010, p. 20). This paper is intended as an initial step towards this goal.

Although the analysis of MES has been ignored in the recent literature, it has not been ignored *tout court*. For the one explicit account of MES, albeit little more than a sketch, one has to go back to a paper by Mark Steiner (1978b). Steiner's core idea is that an MES is a distinctively *mathematical* explanation whose mathematical explanandum happens to be linkable to a real-world physical phenomenon. I think that Steiner's idea, though interesting, is fatally flawed. However, seeing where and how it goes wrong will turn out to shed important light on how a successful analysis of MES might proceed. The key insight, which I will return to discuss in the Conclusion, is that MES's are not really *mathematical* explanations at all, but are better viewed as scientific explanations that appeal to mathematical facts. Hence MES needs to be accommodated by any general account of scientific explanation.

2. The Transmission View

In his influential 1978 paper, 'Mathematics, Explanation, and Scientific Knowledge', Mark Steiner takes up the question of what distinguishes genuine mathematical explanations of physical phenomena from explanations in the physical sciences in which the mathematical apparatus plays other sorts of role. As his principal (and only) case study, he focuses on the Euler rotation theorem, which states that the displacement of a rigid body about a fixed point can always be achieved by rotating the body a certain angle about a fixed axis. Steiner shows how this result, which he describes as having 'profound consequences for mechanics' (1978b, p. 17), can be explained by conceptualizing rotation as a linear transformation of Euclidean space and then looking at the complex conjugates of the eigenvalues of this transformation. Steiner then makes the following claims:

> This is, I stress, a *mathematical* explanation (of a physical fact), though obviously, physical assumptions enter: that physical space is a three-dimensional Euclidean manifold. (Steiner 1978b, p. 18)

> The difference between mathematical and physical explanations of *physical* phenomena is now amenable to analysis. In the former, as in the latter, physical and mathematical truths operate. But only in mathematical explanation is [the following] the case: when we remove the physics, we remain with a mathematical explanation – of a mathematical truth! (Steiner 1978b, p. 19)

Steiner's remarks here are brief, and the rest of his paper is devoted to other issues. As such, it amounts to a sketch of an account of MES rather than a fully articulated analysis. Nonetheless, Steiner's core thesis is clear enough, and can be illustrated schematically by the following diagram, in which P* is some physical phenomenon, M* is a mathematical theorem, and M is a set of basic mathematical axioms and rules:

$$M \rightarrow M^* \text{ -------- } P^*$$

Figure 1: MES of a physical *explanandum*, P*

The basic idea is that a mathematical claim, M*, is linked to the physical *explanandum*, P*, by some sort of mapping relation.[4] What is distinctive here is that the arrow connecting M to M* represents an *explanatory* relation. Crucially, for Steiner, the overall MES (of P*) inherits its explanatory power from the core, purely mathematical explanation (of M* by M) which it contains. Henceforth I shall refer to the general view of MES encapsulated in the above diagram as the *Transmission View* (since it sees explanatoriness as being 'transmitted' from a pure mathematical explanation to the overall MES).

The claim that Steiner makes in the second of the passages quoted above is that a genuine MES is distinguished by the fact that when the right-hand side of Figure 1 is deleted, what remains is a purely mathematical explanation (based on M) of the mathematical truth, M*. This is in contrast to standard scientific explanations, which also typically employ mathematical apparatus, but which are such that – in Steiner's words – 'after deleting the physics nothing [explanatory] remains' (Steiner 1978b, p. 19). I shall refer to this specific claim as *Steiner's Hypothesis*. It seems clear from his remarks that Steiner sees this hypothesis as being an essential component of his view, and hence it ought to be retained as part of the Transmission View of MES that was elaborated above.

One might wonder whether Steiner's talk of 'deleting the physics' really corresponds to excising the right-hand side of Figure 1. In support of this interpretation, consider what Steiner writes about his core example of involving the Euler rotation theorem:

> In our example, the 'bridge' between physics and mathematics is the assumptions that space is three-dimensional Euclidean, and that the rotation of a rigid body around a point generates an orthogonal, real, proper transformation. … Deleting

[4] There is ongoing debate in the philosophy of applied mathematics literature concerning the precise nature of this link. See e.g. Pincock (2007), and also Bueno & Colyvan (forthcoming) for criticism of the standard 'mapping account' of the application of mathematics to the world.

> these assumptions, we obtain an explanatory proof of a theorem concerning transformations and eigenvectors. (Steiner 1978b, p. 19)

According to Steiner's Hypothesis (which I am taking to be an assumption of the Transmission View of MES), if we dig below the surface of a given MES then we will always be able to find a 'pure' mathematical explanation at its core in which both the explanandum and the explanans are mathematical. Thus in the cicada example the key mathematical result is that the lowest common multiple of two numbers is maximal if and only if the numbers are co-prime. Within the overall MES, this theorem about natural numbers is mapped onto the physical situation that involves cicada life-cycle period lengths. But at the core of the MES, according to the Transmission View, lies an explanatory proof of this number-theoretic theorem. Similarly, in the meteorological example, the Borsuk-Ulam theorem is in the first instance not a claim about physical magnitudes on the surface of the Earth but rather the purely topological claim that any continuous function from an *n*-sphere into Euclidean *n*-space maps some pair of antipodal points to the same point. On this view, part of the explanatory force of this MES lies in its incorporating an explanatory proof of this topological result.

The Transmission View has been echoed more recently by several philosophers, including Aidan Lyon and Mark Colyvan, who write – concerning the M* - P* link in Figure 1 – that 'these bridge principles do not seem to do anything more than allow the transmission of the mathematical explanations to the empirical domain' (Lyon and Colyvan 2008, p. 241). Commenting on this passage, Mancosu writes that 'it thus appears that a proper account of explanations in science requires an analysis of mathematical explanations in pure mathematics' (Mancosu 2008b, p. 10). All three of these authors can be seen to be endorsing Steiner's view that each MES inherits its (distinctively mathematical) explanatory credentials from the pure mathematical explanation that forms one component of it.

Because of its status as the sole articulated account of MES, originating in Steiner's 1978 remarks, and its sympathetic echoing by other philosophers working on this issue, the Transmission View can fairly claim to be considered the 'received view' concerning the nature of mathematical explanation in science. As it stands, of course, it is not a complete *analysis* of MES, since it ties MES to 'internal' mathematical explanation and at present there is no generally accepted philosophical analysis of explanation in mathematics. But what is to stop us from accepting the Transmission View, at least in outline?

In the next two sections of the paper, I shall argue that – despite its attractions – the Transmission View is fatally flawed, and hence cannot be the correct account of mathematical explanation in science. I shall raise two separate problems for the Transmission View, each of which I take to be decisive. The first problem is that there are examples of MES that incorporate mathematical results whose best available proofs are not explanatory. The second, more basic problem is that the Transmission View mistakenly presumes that mathematical proof should be taken as part of the explanans of a typical MES. I shall now discuss these two problems in turn.

3. The problem of non-explanatory proof

As already mentioned, one nice feature of the Transmission View of mathematical explanation in science is that it makes a clear prediction, namely that any genuine MES contains a purely mathematical explanation (of a mathematical fact) within it. Unfortunately, this predictive claim – which I have been referring to as Steiner's Hypothesis – is not in general true. One reason why it is not true is that there are examples of MES whose mathematical components lack explanatory proofs. I shall start by illustrating this phenomenon using a case study from biology.

3.1 The Honeycomb Theorem

Why do honeybees build the cells of their honeycombs in the shape of hexagons? Biologists have long hypothesized that the answer has to do with economizing on the amount of wax per unit area. Wax is energetically costly to produce, so it makes sense for bees to use as little as possible when building their combs. As it turns out, it can be proved that the hexagonal tiling of the plane into unit areas is optimal in terms of minimizing the perimeter of the individual cells. This explains why honeybees build hexagonal cells.[5] For clarificatory purposes, I set out one possible regimentation of the explanation below.

(1) Building cells which minimize perimeter per unit area is evolutionarily advantageous (under constraints b_i)

(2) Regular hexagons minimize perimeter per unit area (given assumptions m_i) [geometrical theorem]

(3) Hence cell-building organisms are likely to evolve building techniques which produce hexagonal cells.

(4) Honeybees are limited by biological constraints b_i.

(5) Hence honeybees are likely to produce hexagonal cells.

In order for this to be a counterexample to the Transmission View, two claims about it must be established. Firstly, that it is a genuine MES. Secondly, that the mathematical result at its core is not explanatory. On the first of these points there is not much to be said, other than that biologists do generally take this to be the best explanation of why honeybees build their cells in the shape of hexagons, and that it clearly makes nontrivial use of mathematics.[6]

---

[5] The first discussion of this example in the philosophical literature, as far as I am aware, is in Lyon and Colyvan (2008), although their remarks on it are relatively brief.

[6] It is worth noting that the key theorem involved is geometrical, and that there might turn out to be reasons for treating geometrical explanation in science as its own distinct category of explanation. Nonetheless, since geometry is unquestionably part of mathematics, this feature does not undermine the case for treating the honeycomb example as a genuine MES.

What about the second claim, that the mathematical core of the honeycomb example is not explanatory? Referring back to Figure 1, the issue turns on the explanatoriness or otherwise of the M → M* relation, in other words whether the salient mathematical result, M*, has an explanatory proof. In the honeycomb example, M* corresponds to the following theorem:

Honeycomb Theorem Any partition of the plane into regions of equal area has perimeter at least that of the regular hexagonal honeycomb tiling.

Despite the fact that the general conjecture dates back to antiquity, the above theorem was only proved in full generality by Thomas Hales (2001). What I want to argue is that Hales's proof does not provide an *explanation* of the truth of the theorem.

I shall begin by sketching some of the key ideas involved in Hales's proof.[7] As already mentioned, the basic problem is one of optimization, namely to minimize perimeter for a collection of cells of unit area. One of the major challenges in proving the Honeycomb Theorem is that it is not true locally. In other words, if the challenge is to enclose a single unit of area with the minimum perimeter then the optimal shape is not a hexagon but a circle.[8] Once we shift to consider multiple cells, however, it becomes clear that using circles is disadvantageous because they cannot be fitted together without leaving gaps between the cells. When we are dealing with regular polygons, therefore, local performance (with respect to area to perimeter ratio) can be improved by increasing the number of sides, so that the polygon more closely approximates a circle. However, if we want the polygons to fit together without gaps, then it is a direct consequence of Euler's formula ($v - e + f = 2$) that the average number of sides cannot be more than 6.[9] Hence any polygon with more than 6 sides must be counterbalanced by some other polygon in the tiling that has fewer than 6 sides. Hales's approach is to encapsulate the above insight by introducing into the key optimization equation a penalty term that quantifies the global 'cost' of a polygon having more than 6 sides.

A second way of locally enlarging the area enclosed is by using shapes with curved sides rather than straight sides. Thus, for a given polygon, replacing a given straight side with a convex curved side (i.e. one that bulges out) increases its area to perimeter ratio. However this means that one of its neighbouring polygons must have a corresponding side that is concave, thus reducing its own area to perimeter ratio. Hales therefore adds a second penalty term into the governing equation that represents the global cost of a polygon having curved sides.

The proof then proceeds by verifying that the penalty terms in the optimization equation correctly characterize the effects of changing the shapes of the constituent polygons in various ways, and then by deriving that the regular hexagon is optimal with respect to these penalty terms.

---

[7] The full proof runs to 18 pages, so this will be of necessity no more than a brief overview.

[8] This in known in the mathematics literature as the *isoperimetric problem*.

[9] Since every vertex in a finite graph corresponds to at least three half edges, $e \geq (3/2)v$, so (by substitution into Euler's formula), $(2/3)e - e + f = 2$. Hence $f = (1/3)e + 2$, from which it follows that $e < 3f$. Since an edge borders two faces, the average number of edges per face cannot be greater than 6.

3.2 Why the proof of the Honeycomb Theorem is not explanatory

There are at least four reasons for thinking that Hales's proof, ingenious though it is, does not explain *why* the hexagonal tiling of the plane is optimal. I shall present and discuss these reasons in increasing order of philosophical abstraction.

The first reason (purely circumstantial, but important nonetheless) is that mathematicians working in this area appear not to find Hales's proof especially explanatory. This is manifested both in comments made about the proof, and also in attempts to 'improve' various aspects of it. Some of these improvements have been in minor technical details (for example Frank Morgan has recently shown that one of the lower bounds in Hales's Chordal Isoperimetric Inequality can be raised from $\pi/8$ to $\pi/4$). But there have also been attempts, unsuccessful thus far, to find 'a simpler, more geometric version of [Hales's] proof' (Carroll et al. 2006, p. 1).

Secondly, while there seem to be good motivating reasons for including penalty terms of the sort described in the previous section, these reasons do not show why these penalty terms take the particular values that they do. Consider the full statement of Hales's crucial *Hexagonal Isoperimetric Inequality*:

> Consider a curvilinear planar polygon of N edges, area A, and perimeter P. Let P* denote the perimeter of a regular hexagon of area 1. For each edge, i, let a_i denote how much more area is enclosed than by a straight line. Then
>
> $$P / P^* \geq \min\{A, 1\} - .5\Sigma a_i - .0505/2^4\sqrt{12}(N - 6) \ ,$$
>
> with equality only for the regular hexagon of unit area. (Morgan 2000, pp. 161-2)

Even ignoring the technicalities of the background definitions, it is possible to pick out from the above equation the penalty terms for a polygon having curved sides (as a function of a_i) and for having extra edges (as a function of N). Focusing on the second of these, the specific value of the penalty term is $.0505/2^4\sqrt{12}(N - 6)$. The presence of the (N – 6) term can be explained by reference to the earlier remarks concerning Euler's formula: we know from this that the average number of sides of polygons tiling the plane cannot be greater than 6, hence it is only at this point that the overall penalty term becomes positive. But what about the coefficient $.0505/2^4\sqrt{12}$ on the front of the penalty term? Nothing in the general argument motivating Hales's approach gives any guidance about the specific value of this coefficient.

A third reason for questioning the explanatoriness of Hales's proof is foreshadowed in the remark quoted earlier about searching for a 'simpler, more geometric version' of the proof. The fact is that the bulk of Hales's proof of the Honeycomb Theorem involves not geometry but other quite distinct areas of mathematics such as measure theory and numerical analysis. Implicit in the quoted remark is the assumption that a more geometric proof would also be more explanatory. The point behind this is presumably not that there is anything explanatorily distinctive about geometry per se, but rather that the theorem in question is geometrical and hence it makes sense to favour geometrical reasoning in its proof. In other words, it seems natural to look for geometrical reasons for a result in geometry, number-theoretic reasons for a result in number theory, and so on. The more general claim here is that there is a link between the *purity* of its proof and its explanatoriness. While there are definitely aspects of mathematical practice that support it, as well as suggestive links to the distinction between intrinsic and extrinsic explanations in science, this view is not without its philosophical problems. For one thing, it turns out to be surprisingly difficult to define an appropriate notion of purity that does not fall foul of fairly basic counterexamples (see, for example, Arana 2009). More importantly, it is not even clear that impurity is unequivocally a bad thing when it comes to the explanatoriness of proofs. For example, impure proofs may tie together apparently disparate areas of mathematics in a way that unifies them and provides a more global embedding of the result being proved.[10]

While I am hesitant, therefore, to emphasize impurity as a reason in its own right for questioning the explanatoriness of Hales's proof, the above discussion does pave the way to a fourth and final reason which has considerably more force. In a nutshell, the problem with Hales's proof lies not with its impurity but rather with the specific non-geometrical apparatus that it utilizes. Recall that one of the two approaches mentioned as featuring in the proof is numerical analysis. The overall goal of this subfield of mathematics is the design and evaluation of techniques to give approximate but accurate solutions to 'hard' problems in continuous mathematics, in other words problems for which no closed-form solution is available. In his proof of the Honeycomb Theorem, Hales uses numerical analysis to verify – through brute computation – some of the approximations he makes concerning upper and lower bounds. Indeed the central part of the proof involves various subdivisions into arbitrary-looking special cases. As Carroll puts it, Hales's proof 'becomes a long, arduous case analysis using five separate intermediate lower bounds' (Carroll et al. 2006, p. 7). The two key features here are disjunctiveness and the role of computations. These are both features that philosophers have argued tend to weaken the explanatory power of mathematical proofs (see e.g. Baker 2008). It also seems to fit with our intuitions. In general, if a conjecture is broken down into a large number of separate subcases, each of which is then verified by a distinct computation, this in itself does not give a sense of *why* the conjecture is true.

3.3 Other examples of non-explanatory proofs

Although the honeycomb case study provides a compelling counterexample to the Transmissionist claim that every MES incorporates a purely mathematical explanation at its core, I do not want to rest my entire case against the Transmissionist on this one example. For the purposes of refuting the universal generalization that defines the Transmission View, one definitive counter-instance is of course enough. However, I want to avoid giving the impression that the honeycomb example is an unusual or isolated case. Consider the following example. Why is it that maps showing different countries, states, or provinces can be coloured using no more than four colours in such a way that no two regions of the same colour are contiguous? The explanation is given by the Four-Colour Theorem, which states that given any separation of a plane into contiguous

[10] A more extreme view would be to treat intrinsicality as a virtue of proofs that is distinct from explanatoriness, and to see impurity as generally *increasing* the explanatory power of proofs.

regions, the regions can be coloured using at most four colours so that no two adjacent regions have the same colour. This looks like a genuine MES, yet the Theorem itself – first proved by Appel and Haken in 1978 – is notoriously long and complicated, to the extent that a computer is required to check all of the different subcases. Once again, we have an MES without any explanatory proof 'inside' it.[11] Counterexamples of this sort can, I think, be multiplied. Taken together they cast considerable doubt on the truth of Steiner's Hypothesis, and thus also on the cogency of the Transmission View. In fact, as will become apparent in section 5, very few of the examples of MES that arise in scientific practice conform to the Transmissionist picture.

4. Proof and explanation

4.1 Ways out for the Transmissionist

I have argued that the honeycomb example is a genuine MES, yet it features a mathematical theorem whose best available proof is not explanatory. Faced with this example, there seems to be only two possible responses for the supporter of the Transmission View of MES, depending on which of two bullets she chooses to bite.

One response is to insist that the honeycomb example is *not* a genuine MES. But for this to be a non-question-begging response, the Transmissionist cannot simply rule out the honeycomb case as a genuine MES *in virtue* of it lacking a pure mathematical explanation at its core. The Transmissionist could, however, argue that the honeycomb example does not genuinely explain why bees build hexagonal cells. Without a general account of scientific explanation in place, it is difficult to settle the explanatoriness question definitively. But the Transmissionist line looks unpromising insofar as it flies in the face of scientific practice: biologists discussing the hive-building behaviour of bees certainly talk as if the Honeycomb Theorem provides a satisfying explanation of the hexagonal shape of their cells. Alternatively, the Transmissionist could argue that the honeycomb example is explanatory but that it is not genuinely mathematical. Once again, however, this does not look especially plausible. The overall explanation clearly has an distinctively mathematical component, moreover this component cannot be eliminated from the explanation without fatally weakening it.

A second bullet-biting response for the Transmissionist is to concede that the honeycomb example is a genuine MES but to maintain that the proof of the mathematical component of the overall MES is in fact explanatory. This is the line taken by Aidan Lyon and Mark Colyvan in their discussion of the Honeycomb Theorem. They write

> [Hales's] proof explains why a hexagonal grid is the optimal way to divide a surface up into regions of equal area. So the honeycomb conjecture (now the honeycomb theorem), coupled with the evolutionary part of the explanation, explains why the hive-bee divides the honeycomb up into hexagons rather than

[11] Improvements have been made to the algorithm in Appel and Haken's original proof, for example by Robertson et al. in 1996, but this has not removed the need for computer verification of the proof.

> some other shape, and it is arguably our best explanation for this phenomenon. (Lyon and Colyvan 2008, p. 230)

The direction of this argument, proceeding from the explanatoriness of the mathematical proof to the effectiveness of the overall explanation of the biological phenomenon, exemplifies the Transmission View of MES. I agree with Lyon and Colyvan's conclusion, that the honeycomb conjecture features in a genuine MES, but I think that they are wrong in their assertion that Hales's proof 'explains why a hexagonal grid is … optimal', for the reasons I set out in detail in section 3.2. It is striking that Lyon and Colyvan offer no arguments in support of their claim that Hales's proof is explanatory. This might lead one to believe that they are simply equating proof with mathematical explanation, however in a footnote to the passage quoted above, they write

> Conventional wisdom has it that not all proofs are explanatory; some do and some do not *explain* the theorem in question. The idea is that proof and intra-mathematical explanation can come apart. Interesting as these issues are, we will not pause over them here. (Lyon and Colyvan 2008, p. 230, footnote 3)

It is curious, given these remarks, that Lyon and Colyvan do not say anything more in defence of their claim of the explanatoriness of Hales's proof. In any case, there seems to be nothing in their treatment of the Honeycomb Theorem example that poses a substantive challenge to the arguments I have given.

It is worth mentioning in this context that Steiner has his own analysis of pure mathematical explanation (within mathematics) that is presented in another paper (Steiner 1978a). This opens up the possibility that Steiner could defend his Hypothesis by arguing that the proof of the Honeycomb Theorem does come out as explanatory on his analysis of pure mathematical explanation. I shall not pursue this option, for three reasons. Firstly, as I have argued, there is consensus among working mathematicians that the Honeycomb proof is not explanatory. Secondly, Steiner's analysis of mathematical explanation in mathematics has faced a number of powerful objections in the philosophical literature (see e.g. Hafner and Mancosu 2005). Thirdly, as will be presented in section 5.3, there are examples of MES in which no pure mathematical proof features at all, in which case discussion of the explanatory / non-explanatory proof distinction is bypassed altogether.

4.2 Degrees of generality

There does seem to be something of a puzzle here, which can be encapsulated in the following question: why are MES's – which are (by definition) explanations that make substantive use of mathematics – typically not explanatory 'from a mathematical point of view'? Below I sketch one potential answer to this question.

Philosophical discussions of scientific explanation often emphasize the connection between explanatoriness and *generality*, and this holds equally for philosophical discussions of explanation in mathematics. The basic idea – which gets cashed out differently in different analyses – is that an important way of explaining a given phenomenon is to show how it is deducible from, or caused by, or is a special case of some more general phenomenon, law, or pattern. If there is something to this idea, then it would seem to suggest that the generality that contributes to the explanatoriness of MES's is something that should also make them more explanatory in purely mathematical terms.

To see why the above argument is too quick, we need to consider how generality plays out in some actual examples. So let us return to our two favourite case studies from biology. In the cicada case, the explanation is general in the sense of showing why any periodical organism with periodical predators is likely to evolve a life-cycle period that is prime. In the honeycomb case, the explanatory argument generalizes to cover any situation in which it is evolutionarily advantageous to enclose large numbers of equal areas using a minimum of materials. So these explanations are indeed general, in virtue of potentially applying to a wide range of organisms under a wide range of ecological conditions. However, the generality in question is restricted in various ways – to organisms acted upon by natural selection, given the actual laws of chemistry and physics – and thus, I shall argue, falls short of the kind of generality that is explanatorily relevant in mathematics.

Consider once again the honeycomb case. Faced with the core result of the Honeycomb Theorem, mathematicians are interested in whether – and, if so, how – Hales's proof generalizes to scenarios in which various key assumptions are altered or eliminated altogether. For example, are hexagons optimal for tiling other kinds of surface such as the sphere, or the torus, or the Möbius strip? What if we allow a mixture of two different sizes of cells? What if we assume that the walls of the cells have non-negligible width? And so on. None of these various questions are directly answered by Hales's proof, nor by any simple transformation of it, which is another important reason why mathematicians consider it to be relatively unexplanatory.

Notice, however, that there is no contradiction – nor even any real tension – in the honeycomb example being both biologically general and mathematically non-general. The point is that mathematicians are typically interested in a level of generality that is significantly greater than what is relevant to scientific applications. Hence lacking this degree of generality is no handicap to a given MES being a good scientific explanation, even though it may rule it out as being straightforwardly transformable into a good pure mathematical explanation.

5. Varieties of mathematical explanation in science

5.1 Mathematics-driven explanation in science

It is not part of my aim here to offer a full-fledged positive analysis of mathematical explanation in science. However I do want to take some preliminary steps in that direction, in particular by drawing a distinction that has not previously been explicitly recognized between two importantly different kinds of MES.

The first kind I shall refer to as *mathematics-driven explanation in science* (MDES). Examples of this kind tend to feature some novel, interesting, or unexpected result, M*, in pure mathematics. The striking nature of M* leads mathematicians (and sometimes also philosophers) to go looking for 'applications' of M* to the real world.

Consider, for example, the Stone–Tukey Theorem, which states that given *n* measurable 'objects' (i.e. sets of finite measure) in *n*-dimensional space, it is possible to divide all of them in half according to volume with a single ($n-1$)-dimensional hyperplane. One entertaining application of this result is for the case when $n = 3$ and the three objects of any shape are a chunk of ham and two chunks of bread – notionally, a sandwich – which can then each be bisected with a single cut (i.e. a plane). Indeed for this reason the theorem is often referred to as the Ham Sandwich Theorem. Turning this around, we can cite the Stone-Tukey Theorem in an explanation of why, regardless of the shape of the slices of bread and ham, such a bisection can always be performed. And this explanation looks to be a genuine MES. Hopefully it is fairly obvious why I am referring to cases of the above sort as 'mathematics-driven' MES. The point is that the impetus for constructing such an MES comes from the proof a mathematical theorem and not from some physical phenomenon that puzzles us.

It is with respect to this subcategory, MDES, of MES that the Transmissionist account has the best chance of success. In particular, if the pure mathematical result, M*, has an explanatory proof, then any MDES that is built around it will automatically incorporate this pure mathematical explanation, thus satisfying the key claim of the Transmission View. Unfortunately for the Transmissionist, this subcategory comprises a form of MES that is decidedly marginal on a number of counts.

MDES's are explanatorily marginal, because it is often unclear whether putative cases of MDES really are *explanations* at all. The problem is that, since it is a mathematical result that is the driving force behind the development of a given MDES, the physical phenomenon it gets linked to is often a prediction rather than an explanandum. Thus in the Borsuk-Ulam example discussed earlier, the existence of antipodal points with the same temperature and pressure was not a physical fact whose truth was known prior to the prediction made using this theorem. (For more criticism along these lines, see Baker 2005, pp. 226–7.)

MDES's are also marginal with respect to scientific practice. Although they do forge links between mathematics and physical phenomena, the phenomena that are linked to by MDES's are typically of little scientific significance in their own right. Knowing about the existence of antipodal points of the sort predicted by the Borsuk-Ulam theorem, for example, does not help climate scientists develop better models, or explain existing climate trends.

For these reasons, even if a more limited defence of the Transmission View could be mounted as at least giving a correct account of mathematics-driven explanation in science (MDES), this would be something of a pyrrhic victory. For while there may be some sense in which examples of MDES can be considered 'mathematical explanations in science', both their explanatoriness and their links to science are peripheral at best. It seems clear, therefore, that the Transmissionist analysis fails as a general account of mathematical explanation in science.[12]

[12] Moreover, even within this restricted category of MES, the Transmissionist analysis does not always hold. For there are plenty of interesting pure mathematical results that lack explanatory proofs. Any MES constructed on the basis of such a result will lack a mathematically explanatory core and thus, despite being an MDES, will falsify the Transmissionist claim. Indeed there are good grounds for putting the Borsuk-Ulam Theorem example in this category. Although various

5.2 Science-driven mathematical explanation

The second kind of MES I shall refer to as *science-driven mathematical explanation* (SDME). An MES of this sort has its origin in some puzzling physical phenomenon, or in some striking pattern of such phenomena. In looking for an explanation, it may turn out – often unexpectedly – that a substantive mathematical result is involved. Alternatively, the putatively best explanation of the given phenomenon will involve an as yet unproven mathematical conjecture, in which case the linking of this conjecture to the physical phenomenon may provide extra impetus for mathematicians to try to come up with a proof. Yet another scenario is when the striking physical phenomenon leads to the formulation of a new mathematical conjecture, which may also then be proved. An example of the first sort is the cicada case study. The genesis of this MES lay with biologists' puzzlement concerning the prime-numbered life cycles of the two subspecies of periodical cicada. This was then linked up, via a claim about the evolutionary advantage of avoiding periodical predators, to some basic theorems in number theory concerning Lowest Common Multiples. The honeycomb example falls into the second of the above three subcategories of SDME. Once again, the original impetus for the development of the MES lay with a striking physical pattern, of honeycomb cells in beehives. Unlike in the cicada case, however, the key mathematical claim in the honeycomb case had not been proven at the time when the MES was first put forward. Moreover, Thomas Hales acknowledged that it was its link to this striking biological phenomenon that encouraged him to choose to work on proving the Honeycomb Conjecture (Hales 2001, p. 3).

My claim is that science-driven mathematical explanations (SDME) rarely contain pure mathematical explanations of the mathematical results they utilize, and even when they do it makes no substantive contribution to their overall explanatory power. Hence the Transmission View, which sees all MES's as incorporating a distinctively mathematical form of explanation, fails completely for SDME.

5.3 The problem of excluded proofs

My discussion of the honeycomb case study has hopefully done enough to undermine the initial plausibility of Steiner's Hypothesis, since this hypothesis rules out the possibility of there being MES's whose mathematical components lack explanatory proofs. However, I think that there is also a second, deeper reason why Steiner's Hypothesis is false. Recall that what it claims is that any genuine MES contains an explanatory proof of a pure mathematical result. Thus far I have been focusing on the distinction between explanatory and non-explanatory proofs, arguing that the proof of the Honeycomb Theorem, for example, is not explanatory despite being involved in a genuine MES. But there is a sense in which even formulating this first objection already grants Steiner an important point, namely that a typical MES does include a mathematical proof of some sort. This is an implication of Steiner's Hypothesis that can be separated from the explanatory / non-explanatory proof issue, and I think that it too is false. It is false because the core exemplars of MES, namely the science-driven mathematical

---

proofs have been given of the Borsuk-Ulam Theorem, none of them give a clear intuitive answer to *why* the result holds.

explanations identified in the previous section, do not typically contain any mathematical proofs at all.

In order to illustrate this point, it may be helpful to revisit the cicada case study that was presented in section 1.1. As mentioned above, I think that the cicada case study is a good example of a SDME. Below is the version presented and discussed in Baker 2005:

(1) Having a life-cycle period which minimizes intersection with other (nearby / smaller) periods is evolutionarily advantageous. [**biological law**]

(2) Prime periods minimize intersection (compared to non-prime periods). [**number-theoretic theorem**]

(3) Hence organisms with periodic life-cycles are likely to evolve periods that are prime.

(4) Cicadas in ecosystem-type, E, are limited by biological constraints to periods from 14 to 18 years. [**ecological constraint**]

(5) Hence cicadas in ecosystem-type, E, are likely to evolve 17-year periods.

The above argument is intended to explain both why cicada periods are prime and why cicadas in a particular ecosystem have period lengths of 17 years. It is a mathematical explanation because it makes essential use of the number-theoretic theorem in premise (2). But note that this explanation does *not* include any proof of this mathematical result. I take this to be an absolutely standard feature of SDME's such as this one.

We need to be careful, however, of jumping straight to the conclusion that SDME's do not contain mathematical proofs. After all, even the above argument is not a fully worked-out SDME. More needs to be said (and was said, at some length in Baker 2005) about why minimizing intersection with other nearby periods is evolutionary advantageous. And more also needs to be said about how the ecological constraints act together to limit the range of possible period lengths. Given all this, is it not reasonable to think that the proof of the number-theoretic theorem in premise (2) will also be included in a fully fleshed out version of this SDME?

The above line of objection has some plausibility. Thus, in my presentation of the cicada argument, I do say more about premise (2). In particular I show how it follows from the following two lemmas:

Lemma 1: the lowest common multiple of m and n is maximal if and only if m and n are coprime.

Lemma 2: a number, m, is coprime with each number $n < 2m$, $n \neq m$ if and only if m is prime.

However, the proofs of these two lemmas – while relatively elementary – were not presented or discussed; instead readers were referred, without further comment, to

Edmund Landau's *Elementary Number Theory* (Landau 1958). This suggests that even a more fleshed-out version of the cicada SDME need not contain any mathematical proofs.

The Transmissionist is committed to the thesis that in order to evaluate the explanatory worth of a candidate MES, we need to check whether the proof of the mathematical theorem (or theorems) it invokes is itself explanatory. It is clear that scientists do not generally do this. Moreover, this does not look like mere laziness on their part, nor some sort of implicit 'division of labour' policy, leaving the business of evaluating mathematical explanations to the mathematicians. On the contrary, it seems to be quite rational not to care about whether the proofs of the mathematical results utilized by a given SDME are explanatory. Intuitively, all that matters for the purposes of constructing an adequate SDME is that the mathematical results being used have been proved, because all the scientists are relying on is that these results are demonstrably true.[13]

5.4 Appeal to 'bare' mathematical results

In general, all scientists need to know when they appeal to a given mathematical result in the context of a science-driven mathematical explanation is *that* this result has been proved. Nothing else about the proof matters for the purposes of the overall SDME, nor does the proof need to be included in the presentation of the SDME. Why not? How can these bare mathematical results, these 'unexplained explainers', avoid undermining the acceptability of any SDME in which they appear?

Actually the situation is much less mysterious than it initially appears to be. For it is a relatively familiar point from the literature on scientific explanation (dating back at least to Hempel 1973) that an explanation can be complete even if the facts to which it appeals are left unexplained. This is because a scientific explanation tends to be targeted at answering a specific question, and the context in which this question is posed will determine what can and cannot legitimately be left unexplained. Marc Lange gives the example of explaining the presence of a rainbow, and writes that 'we do not need to know *why* there were water droplets in the air in order to use that fact to explain why the rainbow occurred. We need to know only *that* there were water droplets in the air'. Similarly, 'we do not (in a typical context) have to know why the laws of refraction and reflection hold' (Lange 2002, pp, 99–100). One way of thinking about this example is to distinguish two questions that we might want to answer: 'Why did the rainbow occur?' and 'Why were there water droplets in the air?' Lange's point is that, in most normal contexts, we can fully and satisfactorily answer the first of these questions without answering the second.

How is this relevant to mathematical explanation in science? Consider our main example of a SDME, the honeycomb case. The main question to be answered is why bees build the cells of their hives in the shape of hexagons. An explanation that makes appeal

[13] An interesting question, which I shall not pursue here, is whether it is always *necessary* for a mathematical result appealed to in a SDME to have a formal proof available. Juha Saatsi (in conversation) has argued that we already had enough inductive evidence of the truth of the Honeycomb Conjecture for it to be usable to explain the shape of bees' cells even prior to Hales' formal proof of this result.

to the bare statement of the Honeycomb Theorem seems sufficient to answer this question. It does so by showing that hexagons solve a particular maximization problem (i.e. the isoperimetric problem). However there is also another question in the offing: why do hexagons yield the optimal ratio of perimeter to area when tiling the Euclidean plane? It is this second question for which an explanatory proof of the Honeycomb Theorem would provide an answer. Two points are worth making about this example. Firstly, the initial question is driven by biological concerns while the second question is driven by mathematical concerns. Secondly – and crucially – answering the second question does not seem to add anything to the explanation of the first. This shows, I think, why this SDME – and SDME's in general – do not (and should not) incorporate proofs of the mathematical results to which they appeal.

6. Conclusions

According to the Transmission View, which takes as its starting point the remarks on mathematical explanation in Steiner (1978b), a mathematical explanation in science (MES) is a pure mathematical explanation whose mathematical explanandum happens to be mapped onto some physical phenomenon of interest. This physical phenomenon is then explained by the MES in virtue of the MES incorporating a purely mathematical explanation at its core. In a sense, therefore, the Transmissionist sees an MES as a distinctively mathematical explanation that 'spills over' into empirical science.

My main goal in this paper has been to show that the Transmission View is inadequate. It is inadequate because it fails to cover the science-driven mathematical explanations (SDME's) which are the most numerous, and most paradigmatic, examples of MES. One of the main implications of the Transmission View is Steiner's Hypothesis, according to which any genuine MES must incorporate an explanatory proof of the core mathematical result to which it appeals. My attack on Steiner's Hypothesis has been two-pronged. Firstly, I gave an extended example (involving the Honeycomb Theorem) of an MES whose core result lacks an explanatory proof. Secondly, I argued that – regardless of their explanatoriness – mathematical proofs typically do not feature as part of SDME's.

What are the implications of the failure of the Transmission View? On the one hand, this suggests that MES's are less *mathematically* significant than might at first be thought. A typical MES is not mathematically explanatory, nor does it contain any mathematically explanatory components. On the other hand – and as a direct result of the previous point – this also suggests that MES's are of increased *philosophical* significance. If, as I have argued, MES's are not pure mathematical explanations 'in disguise,' then this leaves just a couple of options for how to classify them. Either they are full-fledged scientific explanations, despite their invoking mathematics. Or they are some third category, distinct both from scientific explanations and from pure mathematical explanations. I take it that – on grounds of parsimony, if nothing else – the latter option should only be taken if all other alternatives fail. Hence we ought to treat MES's as scientific explanations.

If this is right, then MES cannot legitimately be bracketed off from discussions of the best philosophical account of scientific explanation. Mancosu writes that 'one of the major philosophical challenges posed by mathematical explanation of physical phenomena is that they seem to be counterexamples to the causal theory of explanation' (Mancosu 2008a, p. 135). What I hope to have shown is that MES cannot defensibly be lumped together with mathematical explanation within mathematics, and hence the challenge that mathematical explanation in science poses to general accounts of scientific explanation cannot merely be dismissed.[14]

Department of Philosophy
Swarthmore College
500 College Ave.
Swarthmore PA 19081
USA
email: abaker1@swarthmore.edu

References

Arana, Andrew 2009: 'On Formally Measuring and Eliminating Extraneous Notions in Proofs'. *Philosophia Mathematica*, 17, pp. 189–207.

Baker, Alan 2005: 'Are There Genuine Mathematical Explanations of Physical Phenomena?' *Mind*, 114, pp. 223–238.

--------2008: 'Experimental Mathematics'. *Erkenntnis*, 68, pp. 331–344.

Bangu, Sorin 2008: 'Inference to the Best Explanation and Mathematical Realism'. *Synthese*, 160, pp. 13–20.

Batterman, Robert 2002: *The Devil in the Details: Asymptotic Reasoning in Explanation, Reduction, and Emergence*. Oxford: Oxford University Press.

Bueno, Otavio and Mark Colyvan. Forthcoming: 'An Inferential Conception of the Application of Mathematics'. Forthcoming in *Nous*.

Carroll, Colin, Adam Jacob, Conor Quinn, and Robin Walters 2006: 'On Generalizing the Honeycomb Theorem to Compact Hyperbolic Manifolds and the Sphere'. SMALL Geometry Group report, Williams College.

Cellucci, Carlo and Donald Gillies (eds) 2005: *Mathematical Reasoning, Heuristics and the Development of Mathematics*. London: King's College Publications.

Colyvan, Mark 2001: *The Indispensability of Mathematics*. Oxford: Oxford University Press.

--------2002: 'Mathematics and Aesthetic Considerations in Science'. *Mind*, 111, pp. 69–74.

--------'There is No Easy Road to Nominalism'. *Mind*. Advance Access published June 15, 2010, doi: 10.1093/mind/fzq014

Daly, Christopher and Simon Langford 2009: 'Mathematical Explanation and Indispensability Arguments'. *Philosophical Quarterly*, 59, pp. 641–658.

[14] Earlier versions of this paper were presented at Aarhus University, Keio University, the University of Leeds, the University of Liverpool, the University of Oxford, and the University of Pittsburgh. Feedback from audiences at these venues has led to a number of improvements in the final version. I am also grateful to Jacob Busch, Aidan Lyon, and Chris Pincock for helpful comments and discussion.

Hafner, Johannes and Paolo Mancosu 2005: ‘The Varieties of Mathematical Explanation’. In Mancosu, Jorgensen, and Pedersen 2005, pp. 215–250.

Hales, Thomas 2001: ‘The Honeycomb Conjecture’. *Discrete and Computational Geometry*, 25, pp. 1–22.

Hempel, Carl 1973: ‘Science Unlimited?’ *Annals of the Japan Association for the Philosophy of Science*, 4, pp. 181–202.

Kitcher, Philip 1989: ‘Explanatory Unification and the Causal Structure of the World’. In Kitcher and Salmon 1989, pp. 410–505.

Kitcher, Philip and Wesley Salmon (eds) 1989: *Scientific Explanation*. Minneapolis: University of Minnesota Press.

Landau, Edmund 1958: *Elementary Number Theory*. New York, NY: Chelsea.

Lange, Marc 2002: *An Introduction to the Philosophy of Physics: Locality, Fields, Energy and Mass*. Oxford: Blackwell.

Leng, Mary 2005 ‘Mathematical Explanation’. In Cellucci and Gillies 2005, pp. 167–89.

Lyon, Aidan and Mark Colyvan 2008 ‘The Explanatory Power of Phase Spaces’. *Philosophia Mathematica*, 16, pp. 227–43.

Mancosu, Paolo 2008a: ‘Mathematical Explanation: Why it Matters’. In Mancosu 2008, pp. 134–50.

--------2008b: ‘Explanation in Mathematics’. *Stanford Encyclopedia of Philosophy* [online encyclopedia], Fall 2008 Edition, <http://plato.stanford.edu/archives/fall2008/entries/mathematics-explanation/>, accessed 7 January 2011.

Mancosu, Paolo, K. Jorgensen, and S. Pedersen (eds) 2005: *Visualization, Explanation and Reasoning Styles in Mathematics*. Dordrecht: Springer.

Mancosu, Paolo (ed.) 2008: *The Philosophy of Mathematical Practice*. Oxford: Oxford University Press.

Melia, Joseph 2002: ‘Response to Colyvan’. *Mind*, 111, pp. 75–9.

Morrison, Margaret 2000: *Unifying Scientific Theories: Physical Concepts and Mathematical Structures*. Cambridge: Cambridge University Press.

Morgan, Frank 2000: *Geometric Measure Theory: a Beginner’s Guide*. Academic Press.

Pincock, Christopher 2007: ‘A Role for Mathematics in the Physical Sciences’. *Nous*, 41, pp. 253–75.

Saatsi, Juha: ‘The Enhanced Indispensability Argument: Representational versus Explanatory Role of Mathematics in Science’. *The British Journal for the Philosophy of Science*. Advance Access published December 23, 2010, doi: 19.1093/bjps/axq029

Steiner, Mark 1978a: ‘Mathematical Explanation’. *Philosophical Studies*, 34, pp. 135–51.

--------1978b: ‘Mathematics, Explanation, and Scientific Knowledge’. *Nous*, 12, pp. 17–28.

Wilson, Robert and Frank Keil 2000: *Explanation and Cognition*. Cambridge, MA: MIT Press.

Woodward, James 2009: ‘Scientific Explanation’. *Stanford Encyclopedia of Philosophy*. Stanford Encyclopedia of Philosophy [online encyclopedia], Spring 2010 Edition, <http://plato.stanford.edu/archives/spr2010/entries/scientific-explanation/>, accessed 7 January 2011.