Speaker 1: So Professor Francis Su is the Benediktsson-Karwa Professor of Mathematics at Harvey Mudd College, which is one of Claremont Colleges. He received his bachelor's in Math at the University of Texas, and his PhD at Harvard in Math, and he has also taught at Cornell and Caltech. In addition to being a Harvey grad, he's won a number of teaching awards and has been president of the Mathematical Association of America. In addition to doing great research, though, which he'll tell us about today, he's been passionate in communicating and popularizing mathematics. He has a New York Times ... New York Times has covered his work on fair division, which is dividing up rent or cake or things like that. He is interested in song writing, gardening, photography, and theology. So we're very happy to have Professor Su with us today.

Dr. Francis Su: Thank you very much. Thanks for coming back, those of you who were here yesterday. Unlike yesterday's talk, I'll actually be showing you some mathematics on real mathematical stuff. Actually, yesterday's talk was real mathematical stuff, too, but I mean we'll actually be digging into some mathematical ideas today, associated with some of the problems that I've been working with undergraduates on. So you'll see some real work by undergraduates here, in particular Deborah Berg, who was a student at Harvey Mudd many years ago now when this work first originated and collaborator Kathryn Nyman who is a professor at Willamette College, and the president of my college, Maria Klawe, also makes an appearance in this talk. She is a mathematician who's president of Harvey Mudd.

Dr. Francis Su: So here's the basic plan for today. I'm going to first try to motivate for you why it is that math actually has something to say about the social sciences. And so, if there's any takeaway from the talk today, there are two takeaways, one is that math has something interesting to say about how people relate to one another. And then, the other takeaway is the social sciences actually motivate some interesting mathematics. So I hope to convince you of that, regardless of what level you're at. If you're a freshman and you haven't even taken any math, you should still be able to follow most of this talk, all right? That's my goal.

Dr. Francis Su: The second part, so the first part's sort of motivation, second part is actually some interesting math that's involved, the third is a theorem that we proved, and then the fourth is just some generalizations. Some places to take the work. There might be even some questions here for you to think about.

Dr. Francis Su: Okay, so when you think about sets that model preferences, what comes to mind? So when you think about people expressing preferences, what comes to mind? Any thoughts? Things that you think about when you express a preference, for, oh, I don't know, let's say a variety of pizza. There's an example of something you might express a preference over. What's another thing you might express a preference over in your daily life? Yes, I do want real audience participation here. Just call something out.

Audience: How much ice cream you want.

Dr. Francis Su: How much ice cream you want. Yes, good. What else?

Audience: Favorite classes.

Dr. Francis Su: Favorite classes. What else?

Audience: Least favorite classes.

Dr. Francis Su: Okay. What else.

Audience: Paper or plastic?

Dr. Francis Su: Oh, think bigger. What else?

Audience: [inaudible 00:03:49].

Dr. Francis Su: What's that?

Audience: Presidents?

Dr. Francis Su: Presidents, yes. So yes. Election of presidents or people to congress. That's a big example, a big topic. And if I asked you, then, to think about sets that model preferences, you might think of a picture like this, which appeared a lot in the 2016 election. So here is an example of ways we think about sets that model preferences. Here we have a set of states, red and blue, okay? These are two sets. One of them you might call the Clinton Set, the other you might call the Trump Set. Okay? The set of states that expressed preferences for particular candidates in the last election. All right. There's one example.

Dr. Francis Su: Here's another example. If you are interested in stuff to eat, which it sounded like was on your mind, you might express preferences over pieces of cake, for instance. This is another problem, which I won't talk about in this talk, but it's the problem of fair division. How you divide cake fairly among several people. And I just want to point out here, if I divide the cake somewhere, I could ask someone like Joy. Joy, which piece do you prefer? Piece one or piece two?

Joy: Piece two.

Dr. Francis Su: Piece two. And Joy, why would you prefer piece two?

Joy: Because I like strawberries.

Dr. Francis Su: Because you like cherries. Because you like cherries, apparently. All right, great. Yeah. And so, one of the questions you might ask is, "Is there always a way to divide the cake so that, for instance, two people might want different pieces, yes? And if I asked, Joy said she wants this piece, right? And, Reese...

Reese: Yes.

Dr. Francis Su: ... says he wants, which piece?

Reese: I could go for piece one.

Dr. Francis Su: Okay, so if he went for piece one, you'd have a division where they both prefer different pieces, right? But it's possible that Reese, even though he has different preferences, it's not that he likes cherries, he just likes the larger piece, then this division would not induce them to choose different pieces, and I might want to choose a different division, yes?

Dr. Francis Su: So if I, for instance, divide the cake over here, they'd both want piece two, on the right. And if I divide the cake here, they'd both want piece one, on the left, yes? Okay, so somewhere along the way you might imagine that one of them might change their mind about which piece, right? If I move the knife from left to right, at some point, maybe, one of them would switch preferences. At that point, you'd actually have a fair division. You with me? Okay? And so, here, what do we have? You might mark all the places of the knife where Joy wants piece two, she still might want piece two here because it has lots of cherries, yes? But then at some point she's going to switch her preference. You might also mark all the places where she prefers piece one. Those are the two sets that express her preference. We haven't brought Reese into the picture at all at this point. Yes? Okay, so that's another way to think about sets that model preferences.

Dr. Francis Su: Here's another example. How many people here live with other people? A bunch of people, okay, good. You have housemates or roommates or something like that. And if you live in a house, for instance, one of the things that often happens is you might fight over where to set the thermostat, right? Like, some people run hot, some people run cold. So you might, then, want to understand what preferences people have. And so, if I asked Reese, "Which set of temperatures are you comfortable with?" This is obviously winter setting. He might say, "Oh, I'm happy with the thermostat anywhere in here." Are you with me? Because that's an example of another set that models preferences.

Dr. Francis Su: Okay, so, what do we have here? We have a situation that's often a kind of social situation with some interaction. The mathematics of modeling social interactions is called game theory. And here we have very naturally sets that say something about what people prefer. Okay? And what I hope to convince you of is that, in fact, in each of these kinds of situations, often what's important is the intersection of sets, right? So for instance, if I wanted to make more than just one person happy, if I wanted to make... remind me your name?

Zahara: Zahara.

Dr. Francis Su: Zahara. If I wanted to make Zahara happy, and Reese happy, and Joy happy, they would all have different sets here, yes? And where do you think I should set the temperature, if I have a bunch of sets? Well, it'd be nice if I could set the temperature to a point that's in all the sets? Yes? That's an intersection of sets, right? Aha. And if you think about the cake-cutting example, where do you think I want to cut the cake to make someone indifferent? I'd probably want to cut the cake at a place where the sets intersect, right? So back here, for instance, where she prefers piece one and piece two. The place where it intersects is actually the important piece, right?

Dr. Francis Su: Okay, so intersections are important. The other thing I want you to notice is that in each of these cases, you have some natural notion of closeness. Geometry, typology is often the way we think about this. You can talk about close temperatures. You can talk about close cuts of cake. You can talk about close states, geographically. Okay? And the main idea behind the mathemous talk is just the insight that somehow, this notion of closeness, the typology on this space, actually governs, constrains, the way people behave, in ways they may not even realize. Okay? So just to convince you of that, I want to do a little demonstration.

Dr. Francis Su: Imagine after this talk we all go out to dinner to some restaurant. Okay? Maybe it's a restaurant on, is it Dartmouth Avenue? There's probably no restaurants on Dartmouth Avenue, are there? Baltimore Pike, yes. Okay? Okay, good. So suppose there are 14 restaurants, lots of choices there where we could go to dinner. And we want to decide where are we all going to go to dinner. And generally, you try to make the most number of people happy, yes? So suppose I ask each one of you to pick five restaurants that you'll be okay with going to. If I ask everybody to pick 5 restaurants out of 14, you could imagine that each of you have lots of different preferences, and you might spread out your preferences over these 14 restaurants. And if that's the case, then, if I wanted to predict how popular is the most popular restaurant, might be the case that each restaurant gets an equal number of votes, 5 out of 14. Yes? About 1/3. With me? Okay. But, suppose we actually had a constraint. Suppose it's actually the case that... tell me your name...

Ryan: Brian.

Dr. Francis Su: Brian. Suppose it's actually the case that Brian has some constraint on his preference. Like, maybe he lives near restaurant J, and he's lazy and he doesn't want to walk very far. So he's only willing to pick restaurants that are actually next to each other has his choices, yes? That's a geometric constraint on his preference, yes? His preference set. Suppose all of us had that geometric constraint. Suppose it's a case that supposed all of you, for some natural reason, your preference is picking five restaurants that are next to each other. Consecutive restaurants. Let's observe what happens to the most popular restaurant, how many votes it gets.

Dr. Francis Su: So here's what I want you to do. Each person, I'd like you to pick 5 of these 14 restaurants, and they all have to be next each other, okay? So ABCDE could be one choice, E through I could be another, I through M, et cetera. Just pick five consecutive restaurants. Are you with me? Understand what we are doing? Okay? And this is a line here, so A and N are not next to each other, okay? It doesn't circle around. With me? Okay, good. Everybody now, pick five consecutive restaurants in your mind and raise your hand when you've done it. I want everybody to do this.

Dr. Francis Su: Picked five restaurants? Okay, good. Put your hands down. And now, we're just going to observe, how popular is the most popular restaurant when I call out the restaurants in order. Okay? So I'm just going to see, we'll just look around the room and see... now we know it probably, the most popular restaurant's going to have at least a third of the votes, but let's see how popular the most popular restaurant is. Okay, how many people here voted for A, raise your hand? What about B? Okay, if your hand was up for A, it should still be up for B. Because five consecutive... Yes. Okay, good. Okay, what about C? D? E? F? G? H? I? J? K? L? M? N?

Dr. Francis Su: Okay, what was the most popular restaurant? It looked like I or J, maybe? If you voted for I, raise your hand. What fraction of the room is that? It's a lot, yes? I don't know, maybe 60%, would you say? That's significantly more than a third, isn't it? And I claimed, it had to be that way. In fact, I knew that, because before this talk I made a prediction that, in fact, the most popular restaurant would get at least half the votes. Okay? And so now the rest of the talk I'm just going to explain why I was able to make this prediction. All right?

Dr. Francis Su: All right, so that's kind of interesting. The math, the underlying geometry here of a constraint, the outcome. All right, so to motivate, then, the next part of this talk, I want to talk about elections. Because that's, in fact, what we're here to do, is think a little bit about voting. And here's an episode of The Simpsons from just before an election, and I'm wondering if anybody here knows which election this was? Some of you weren't alive, or didn't exist yet, but, maybe you can guess. Anybody? Students only. Anyone? Yes.

Audience: Nixon?

Dr. Francis Su: No, not Nixon. Yes.

Audience: Bill Clinton?

Dr. Francis Su: Clinton, yes. This is Clinton on the right, and Clinton was up against another person, Senator Bob Dole, okay? And the Simpsons had a very funny episode about this, right before the election, and it was also... was it? Yeah, it was the Halloween episode, which is right before the election... and here's what happened. Basically, some space aliens abducted Clinton and Dole and took them up in a spaceship. Homer's trying to rescue them. And what's actually happened, then, is some aliens have produced body doubles of the candidates, and they're actually the ones running for president, and nobody knows this. Okay? So here's what's... let's just watch here, what happens when everything goes wrong. Let's see if I can do this. Yes. Okay. They're abducted right now, and Homer's trying to rescue them. They're on a spaceship.

Video: Homer, shoot. Oh. Oh no, what have I done? What am I doing? What will I do?

Video: The politics of failure have failed. We need to make them work again. Tomorrow, when you are sealed in the voting cubicle, vote for me, Senator Bob Dole.

Video: I am looking forward to an orderly election tomorrow, which will eliminate the need for a violent blood bath.

Video: America, take a good look at your beloved candidates. They are nothing but hideous space reptiles.

Video: It's true, we are aliens. But what are you going to do about it? It's a two party system. You will have to vote for one of us.

Video: He's right. This is a two party system.

Video: Well, I believe I'll vote for a third party candidate.

Video: Go ahead, throw your vote away.

Dr. Francis Su: Okay, so what commentary were the Simpsons making about elections here? Anybody... yes.

Audience: The fact that we, as a society, will collectively vote for one of the two major parties. And even if the candidate from the third party's great, we tend to sway away because we don't think they have a realistic chance of winning.

Dr. Francis Su: Yes. Okay, good. So this brings up one problem with our current way of voting, is that it induces people to vote against their true wishes. Right? Like, if you truly thought the third party candidate was great, but you know they'd never win, you're essentially throwing your vote away by voting for that person. Okay, so that's one of the problems with the way we currently vote, which is a plurality voting system. And so, mathematicians and political scientists have been interested in studying other voting systems that have better properties. And one of the better properties is... Sorry, one of the other ways of thinking about voting is what's called approval voting.

Dr. Francis Su: Let me tell you what approval voting is. It's basically a method of voting where you can look at a list of candidates like this, and you can vote for as many people as you want. Okay? So it's not one person you vote for. So if I, for instance, voted for just one person, and each person in this room just voted for one person, plurality voting would say, somebody could win with just 10% of the vote, right? If that was the person who got the most number of votes. Are you with me? And that doesn't seem like a very good system, especially for picking a primary candidate, does it?

Dr. Francis Su: On the other hand, approval voting would say I could vote for him, and her, and him, et cetera. As many as I like. Now, of course, if you vote for none of them, you're not really expressing a preference, and if you vote for all of them, you're not really expressing a preference, either. Okay? But it's basically, for me, people who I think are above the bar in some sense. Yes? That'd be one way to think about it. Okay, so, approval voting, what's nifty about it is, it has some good properties. For instance, if I like the third party candidate, I could also still vote for one of the other candidates I thought was still acceptable to me. And so, it has a tendency to encourage sincere voting, rather than strategic voting. It induces people to vote truthfully. All right. Approval voting actually isn't used, as far as I know, in any major country election. But it is used, the American mathematical Society and the Mathematical Association of America both use approval voting in their selections for candidates for election.

Dr. Francis Su: Okay, so, what I'd like to do is just to tell you some of the things we've proved about approval voting. Okay, so approval voting, here's the model that we'll think of. Well, think about candidates are rated on a spectrum, with liberals on the left and conservatives on the right. Okay, so think about it, just a one-dimensional line. But later on we'll change the spectrum in interesting ways. And each person is going to express a preference for... maybe their ideal position on this spectrum is right here. But maybe they're willing to vote for anybody who's close to their position. Okay? So you might imagine, I might have very narrow preferences and only be willing to vote for a small swath here. But somebody might be really happy with a larger set of people. Are you with me? Okay.

Dr. Francis Su: So this is the model. Very simple model, and so you might imagine, then, a whole collection of people, which I will call a society. Mathematicians like to make definitions, and so here's a definition. I'm going to redefine the word society to mean a whole collection of voters, together with... First of all, there's a political spectrum, then there's a collection of voters, and a collection of their approval sets. Okay? So the approval set is the positions on the spectrum they're willing to approve. Are you with me? Okay. So this is a society of five voters, and I just colored them in different colors, here. Okay?

Dr. Francis Su: All right. When you think of a society, think of this picture. And so, now the question I want to ask is, given a society, can I make a prediction about how many votes the winner of the approval election will get? Okay? Because, it's a little unsatisfactory if you have an approval winner who doesn't get a majority of the votes, right? So that's part of the question that motivates this. How popular is the most popular candidate going to be? Think of the point on the spectrum as a platform, and I think of a candidate on the spectrum that's a candidate the person marked. How popular is the most popular candidate going to be? The approval winner?

Dr. Francis Su: Okay, so this brings me to the second part of the talk, which is, what does math have to say about this? Because, wouldn't you agree, if I ask you this question, "How many votes is the approval winner going to get?" You're going to say what?

Audience: Three.

Dr. Francis Su: Three? There are two people here... I'm sorry. There are two candidates that would win at here, if this were the platform. But if this is the platform here, you'd get three votes. Yes? Like, if I knew these were the voters and I were a candidate, I'd probably want to place myself inside as many sets as possible, yes? So I'm looking where these sets intersect. Yes? Okay, good. So here's the question. This is really a question about intersecting sets. Okay, so, let's see some math that might help us talk about intersection.

Dr. Francis Su: So, approval sets are intervals, which means that they're convex, so I'll tell you what a convex set is. So here's a theorem about intersecting convex sets. It's about a hundred years old now, it's called Helly's Theorem. It says that if you have a finite family of convex sets... I'll tell you what that means in a minute... in D dimensions, then if some special condition holds, then all the sets will have a point in common. All right?

Dr. Francis Su: So let me tell you now what convex means. So convex means, a set's said to be convex if it's... in some sense, it looks round. Okay? That's the way to think about it intuitively. But a set is convex if, for instance... here's a convex set... if for instance, anytime I pick two points in the set, the whole line between is also in the set. Okay? So, for instance, here's a set that's not convex, it's a banana shaped set. If I pick two points in the green set, the line between them leaves the green set. Are you with me? That's what it means to be convex. So another way to think about convex is, if I'm standing at one point in a convex room, and there's another point in the room, I can see that point in the room. Okay? So these sets have to be blob-like, they can't have holes, for instance.

Dr. Francis Su: Okay, so what does the theorem say? The theorem says if you have a convex set in D dimensional space, a set of them, such that every D+1 of them intersect, then all the sets must intersect. So does that mean, in two dimensions, if you have a family convex set in two dimensions, if every what?

Audience: Three.

Dr. Francis Su: Three intersect mutually, then all the sets have a point in common. So here is a collection of four sets that are convex, and every three of them intersect. So for instance, here are two sets and here's a third. Would you agree that these two horizontal sets and this vertical set intersect here? It's a triple point, yes? And these two intersect here, yes? Okay. There's lots of triple points, and the claim is there's now a point in all the sets. Do you see one? Yes? Where?

Audience: [inaudible 00:26:35].

Dr. Francis Su: The yellow one, yes? Okay. Now, here's what I want you to realize. This is for four sets, but even if you had 17 million sets, where every triple had a point in common, that means there's a point in all 17 million sets. What? Really? It's a little surprising. So let's see why this is true. First of all, notice that this is not true for this collection of four bananas, because these points are triple points for three of them, if you take red and the top and bottom, that's a triple point. These are all triple points, but is there a point in all bananas? Banana-shaped sets. No. Okay, so, convexity is important.

Dr. Francis Su: Now I'm going to give you an argument for why this might be true, for four sets. Okay? We'll see how to generalize that in a minute. So what would you do here? Well, first of all, would you agree that this point is in both horizontal sets? And would you agree this one is in both horizontal sets? Therefore, the line between them must be in both horizontal sets, because both are convex. Are you with me? Aha, I just used the convexity property. Would you agree these two points are both vertical sets? So the line between them is in both vertical sets, yes? So this line, very conveniently intersects this line, yes? So the point at the intersection of these two lines is in all four sets. You with me? Okay, so I've just shown you why this is true for this particular set of four sets, and you're wondering, "What about in general?"

Dr. Francis Su: Okay, so here's what's kind of cool. It's kind of amazing that if you take, if you have four points in the plane, it's always going to be the case that you'll be able to create either a cross construction with those four points, just like I've done, or you have a triangle with three of them and the fourth point in the middle. Okay? It turns out that any four points you do will have to have one of those two configurations. And the reason for that is basically due to some linear algebra, for those of you who've taken linear algebra. It's basically related to the fact that if you have three directions in a plane, they must be linearly dependent. If that meant something to you, then that's great. If it didn't, don't worry about it, okay? But trust me then, that you have some special stuff going on here.

Dr. Francis Su: So the same things going to be true with 17 million triple points, or more. Right? That they're going to be able to find some way of doing something similar here. Okay? So that's the basic idea of Helly's Theorem. It's basically using some linear algebra, and it says something really strong.

Dr. Francis Su: All right, this will only mean something to those of you who've had analysis. And otherwise, don't worry about it. Okay, so what does this have to do with voting? Well, what's interesting is I didn't learn this theorem until only a handful of years ago. But when I came across this problem related to voting theory, I said, "Gosh, what theorems exist out there?" And I found this one from a hundred years ago, and I said, "Oh, it actually has an implication for voting." Why? Well, if you think about this in one dimension, it says if every pair of sets intersect neutrally, then all the sets have a point in common. So, for instance, here is... What's a convex set on the line? It's just an interval. And it says that if you have a situation where every pair intersect, then they must all intersect. So here, for instance, there are three sets, do every pair intersect? No. So you can't expect Helly to be true, and there's no point in all the sets.

Dr. Francis Su: Here you have, again, three sets, but one of them is not convex, because it has a hole in it. Every pair intersects, but there isn't a point in all the sets, because the set's not convex. Yes? And then in the last case, you do have every pair intersecting. And is there a point in all the sets? Aha. Yes. Okay. And so, Helly's Theorem on a line says, in fact, any pairwise intersecting family of intervals must have a point in all the sets. Whether it's 3 sets or 17 million. Okay? Okay. So what does this mean? It means if you live in a house, whether you have 3 housemates or 17 housemates, if any two people can agree on a temperature, then there should be a temperature that makes everybody happy. Whoa. That's a strong conclusion. If every pair of people can agree on temperatures, the temperature makes everybody happy.

Dr. Francis Su: Let me show you a proof. This is actually one of the things I like to do with any research experience is to give my students problems that they don't know have already been solved. So she didn't know about Helly's Theorem. I just said, "Come up with a proof of this fact." Here's what she came up with. It's really nice. Just a three line proof. So here's what she said. "If everybody has an interval of temperatures, then they have a low temperature and a high temperature in that bracket that they're in." Yes? Okay? "And if every pair of people overlap, then it's always the case that any one person's low must be less than another person's high, because every pair overlap." So this person's low is less than this person's high, yes? No matter how you do this. Oh, great. Well then, just take the maximum of the lows and the minimum of the highs, these are achieved by particular people, because there's only a finite number of people, and it's true for those two people, as well. And so, any temperature in between them will work, is a point that's in all the sets. Ah, pretty nifty. That was very nice. This is an example of what's called the mini-max argument, but Deborah discovered this for herself.

Dr. Francis Su: Okay. So something interesting about voting... and so, let me just say a little bit about the way we're going to think about this now. So, I'm going to call society linear if it's... Political spectrum is a line, which is the way you normally think of it. But later on, we'll talk about other kinds of societies. And if you have this very strong condition, that every pair of people agree on a candidate, or a temperature, or whatever, that's very strong. So we're going to call it super agreeable society, if every two voters have overlapping approval sets. And so, then Helly's Theorem just says the following. It says, "In a super agreeable linear society, there's a candidate that'll make everybody happy." So that's Helly's Theorem.

Dr. Francis Su: And it has this interesting voting application, but it's not so useful, because, first of all, it's a very strong hypothesis, to say that every two people in this room will agree on some candidate, yes? But it's also a very strong conclusion. We don't usually care about making everybody happy. We just want to make at least half the people happy. So this was the beginning of their research question. Is it possible, then, to have a less strong condition imply that half the voters will approve? Are you with me? That was the beginning of the research question.

Dr. Francis Su: And so, here's one thing we came up with. We said, "Let's weaken the hypothesis. Let's say, instead of demanding that every pair of people agree, let's just demand that, among any three people, some pair agree." Okay? So we'll call that an agreeable society. And now, here's the claim. I claim, now, that there's a point in at least half the sets. Okay? And here's a very nice argument that'll justify this. Let's see. Here's a society of voters, yes? So what I'm going to do, is I'm going to look at the left endpoint of all the voters, and look for the left endpoint that's farthest to the right. So here's the left endpoint of these intervals, and this ones farthest to the right. Are you with me? That's the rightmost left point. Now look at the right endpoints of all these sets, and look at the one farthest to the left. Yes? Those are achieved by two voters, I'll call them the green voters. Okay? So, if the two green voters are actually like this, then we have Helly's condition, all the sets will intersect.

Dr. Francis Su: Otherwise, they're like this. And if they're like this, then would you agree that all the other voters, which are blue, will have to overlap one of these? Because, if you take these two green voters and any third blue voter, the agreeable condition says some pair must overlap, and it's not the green pair. So green and blue, or this green and blue, must overlap. Aha. Good. So what that means is, if you're any other blue voter, either you intersect this green one here, or you intersect this green one here, and there's only two options, and so every voter must appear either here or here, so one of these endpoints is in half the sets or more. Okay? It's a version of the pigeonhole principle, if you've heard that. Oh, nifty. Great.

Dr. Francis Su: In fact, this explains what happened with the restaurants. How was I able to make that prediction about the restaurants? I asked each of you to pick how many restaurants?

Audience: Five.

Dr. Francis Su: Five. And how many total restaurants were there?

Audience: Fourteen.

Dr. Francis Su: Fourteen. So if I look at any three of you, would you agree that if three of you picked fifteen restaurants and there are only fourteen restaurants, then some pair of you must have overlapped? Aha. And so, I knew that, in fact, the most popular, the approval winner, would have gotten at least half the votes. All right? Ah, nifty.

Dr. Francis Su: Okay. But this was just the beginning. We can say more. So here's the theorem that we proved. We call the society KM, K out of M, agreeable. If among any M people some K of them overlap, agree on a platform, or a candidate. Okay? And then the theorem that we proved says that in a KM agreeable linear society, there's a candidate that gets a certain fraction of the votes. And that fraction's pretty simple if you know K and M. It's K minus one over M minus one. So particular, in the two out of two case, which is pair-wise intersection, you get the fraction is one over one. You get platform winning all the votes. If you're two out of three, then this fraction is one over two. So if you have a society, this KM agreeable society, there's a platform getting half the votes. And if it's three out of four, then you know some platform wins 2/3 of the votes, which is also another interesting number.

Dr. Francis Su: Okay, so how do we prove this? Now I'm going to just give you a sense of how we proved it. Oh, here's some pictures. So here's a two out of three society. Among every three voters, some pair overlap. And do you see a point in at least half the sets? Make a noise when we get there.

Audience: [inaudible 00:38:56].

Dr. Francis Su: Yeah, there we go, great. Excellent. This is among every four voters, some pair overlap, and there is a point, lots of points, in a third of the sets. And here is among every four voters, some three of them overlap, and there is a point in 2/3 of the sets, okay? Just to convince ourselves that this actually is true in some examples.

Dr. Francis Su: All right, so here's our proof. Again, I'm just going to hand wave a little bit and just give you a sense. And one of the things you realize when you go to a math talk is you don't have to understand all the details, you're just following the general ideas. Okay? So that's all I'm going to ask you to do here. So here's what we do. We basically turn every voter into a dot, and we connect two dots, like dot one and dot three, if voter one and voter three overlap somewhere. Are you with me? Okay, so now, this picture turns into a picture of dots and edges. This is sometimes called a graph in mathematics. And now I turned this problem into this one, yes? Now, why do we do that in math? Why do we turn one problem into another problem? We hope the second problem is ...

Audience: Easier.

Dr. Francis Su: Easier. Or easier to understand, or solve. Okay. So, here's what I want you to do. So what are we looking for? We're looking for places of high intersection. Yes? Tell me how you're going to tell from this picture that there are places of high intersection above. If you couldn't see that top picture, how would you be able to tell here? Take a moment and discuss that with your neighbor.

Dr. Francis Su: Okay? I'd love to hear your thoughts. If you can't see the upstairs picture, you only saw the downstairs picture, would you be able to tell who was involved in overlapping sets with a high number of overlaps? How would you be able to tell? Anyone? Yes, Josh?

Josh: Intersecting lines.

Dr. Francis Su: What's that?

Josh: Intersecting lines.

Dr. Francis Su: What do you mean by intersecting lines?

Josh: Looking for where lines have to cross over because there's no other path to get from that point to that point.

Dr. Francis Su: Oh, you mean here. Like, there's a pair of things that overlap?

Josh: Mm-hmm (affirmative).

Dr. Francis Su: Oh, interesting. Okay. Now of course if you only saw this, that would say that five and six overlap, and four and seven overlap, and five and six could be over here and four and seven could be over here. So, what else might you look for? That's a good place to start, maybe. Yes.

Audience: Like, look for the largest cluster of points that are all connected.

Dr. Francis Su: Oh, so you're demanding, here's a cluster of points where they're, not just these two are connected, but all possible pairs? Is that what you're saying?

Audience: Yeah.

Dr. Francis Su: Aha. Interesting. So when you have a condition, and there's a special name in that, we call this a clique, like a clique of people who all know each other. Clique? I think in graph theory it's called clique for some reason. I don't exactly know why, because it's the same word. It's clique and clique. Okay. But, you're saying that a clique, a clique, would correspond to a place of high intersections. Why? Why is it if every pair in this group of four people overlap, there must be a point in all four sets? Well, because if you ignore these people, this is just Helly's Theorem, isn't it? Pairwise overlap means a point in all the sets. Are you with me? So because of Helly, we see that a clique downstairs corresponds to what we're looking for upstairs. Points of high intersection.

Dr. Francis Su: Okay, so that's great. So the first insight is cliques correspond to intersections, all right? The second thing, then, is we're looking for large cliques. Okay, so, what we are going to do next is we're going to color every dot, every voter, a different color if they overlap. Okay? So for instance, I might color this green and this red, but the condition is, if you have two dots that are connected by a line, you can't use the same color. Are you with me? Now why would I do that? Graph theorists love to color because coloring actually brings out, tells us something about what's going on with the connections. Okay?

Dr. Francis Su: So how many colors will I need if I use this condition... color overlapping, voters different colors... how many colors will I need to color this clique? Four. Okay. So the number of colors you require to do this is going to be related to the size of the clique. Yes? Ah, so that turned a clique problem into a coloring problem. Are you with me? Ah, that's interesting, because graph theorists know lots of things about coloring problems. All right. Now, one thing you might think you need, you might need five colors. You could imagine there's something going on over here that requires, you can't just get away with four. Yes? But you know you need at least four because of a clique of size four, yes?

Dr. Francis Su: Okay, it turns out that getting a graph like this from an agreement situation, it turns out, has this property called being perfect. Which means that the coloring number is actually equal to the clique number. That is, the number of colors it takes to color is actually equal and never larger. I'm not going to prove that for you, but turns out that's a fact, a property of these things. And so you really have turned this problem into a coloring problem. And then finally, I now just have to convince you that this requires lots of colors. Okay? So here's what I'll do. I'll just do it in the two out of three case.

Dr. Francis Su: So if, among every three voters, some pair of you overlap, could you all be colored with just two colors? Or could you all be colored with just two colors? Could you be colored with three colors? Could you all be colored with just two colors? Well, let's see. Sorry. Could you all be colored with just one color, that's what I wanted to say. Could you all have the same color? No, because some pair of you overlap, and you have to be colored different colors, yes?

Dr. Francis Su: So then, because of the two out of three condition, we know that we need to use at least two colors, right? And it turns out that the... in fact, what that means is, there can't be more than the color classes themselves. So the number of people colored red can't be more than two people, can it? Because if there were three people colored red, then you could choose those three people and some pair must have overlapped. Yes? Aha. So if every color is, at most, used by two people, then you have to have lots of colors, yes? And that's basically what's happening in the general K out of M condition. Okay?

Dr. Francis Su: Okay, great. So let me just finish by just giving you a brief sense of some of the ways we've generalized this theorem in the years following. So one way to generalize this is changing the spaces that we've considered. So, instead of one-dimensional spectrum, I have a two-dimensional spectrum. Here is the candidates from the 2008 election, all arrayed on left to right. This is the traditional liberal conservative, and this is related to whether you are more authoritarian or more libertarian. Okay, it's a two-dimensional spectrum. Every candidate's here, and I know you're wondering why all the candidates are in the top right corner, and that's because the political scientist who drew this is European, and to them, all our candidates look conservative.

Dr. Francis Su: Okay, but what's a condition you might expect about a person's approval status. If we took one person's approval status, like if I know that Zahara likes John McCain, and likes Mike Huckabee, then maybe they'll like everybody in between. Yes? That's convexity, isn't it? Aha. Okay. Here's another condition you might think. Like, if you believe these issues are independent, then if you like Bill Richardson, and Barak Obama, then maybe it's not just that you'd like everybody in between, but you'd like everybody who's in between on both axis. And so that would mean that if you like these two people, you'd like the entire box that they span. Okay? That's another condition. It's called box convexity.

Dr. Francis Su: And so, when you study some of these things, it turns out if the sets themselves are actual true boxes, then they have the Helly property, pairwise intersection means there's a point in all the boxes. And so, we have some results. I'm not going to explain these results, except to say, that if you have convexity, our results are pretty weak. You can't guarantee very much, even with a three out of four society. But if you have a box society, then you can guarantee a lot, suddenly becomes a lot stronger. There's a lot of overlap that you know in an approval voting situation.

Dr. Francis Su: So now a good question is what happens with box convexity. This is for true boxes, we don't know what happens with box convexity. Some of you might find that an interesting problem to think about. Here's another spectrum. This is a traditional political spectrum, but you imagine the people out here and the people out here are so zany that they look alike. Okay, so we've made this a circle. You might ask yourself, given a collection of sets on a circle, if every pair intersect, must they all intersect? And the answer is, no. These are four sets, that all pairwise intersect but they don't all intersect. This is kind of like the banana example, isn't it? But, you actually can say that some fraction overlap. That it turns out there's a point in a strict majority of the sets. You can prove that. That's one of the things we prove.

Dr. Francis Su: This is with another former undergraduate. In a circular society, if any two voters agree on a candidate, there's a candidate that gets strict majority approval. Someone saw me give a version of this talk a few years ago and asked me, "Is there a K out of M version?" And I said, "I don't know." And they went home, and a few months later they wrote me back and said, "I proved the K out of M version." This is kind of nice. In a K out of M agreeable circular society, there's another fraction we can guarantee, and it looks like the linear result, except their fraction denominator is slightly bigger, which means the fraction is slightly smaller, which is really nice. And it took us several years to write up our paper, and this person took a few months, and so, what was nice was that these two results got published back to back, in the same journal.

Dr. Francis Su: And then lastly, here's a theorem that we thought about more recently. I've given this talk at some political science conferences, and the first thing that happens is they ask completely different questions. And so, here's the question they asked. They said, "Well, you know this interval assumption seems really restrictive. Because maybe I like this candidate and some candidate over here, but I don't like this person in the middle, the person [inaudible 00:50:54]. Okay? Ah. And so we thought, "Ah, okay." We said, "Well, what if you allow one gap. What fraction can you guarantee with pairwise intersection?" So here's one gap in every colored set, and every pair of intersects, but there's not a point in all the sets. And so here's a theorem that we proved. That, in fact, you can guarantee at least 26% approval for the most popular candidate.

Dr. Francis Su: But in every example we tried, we actually could get a third. So that's a conjecture. We don't know why a third is always possible. This was an example, in fact, where I was working with Katheryn Nyman, and this is another student, Jacob Scott, and I just mentioned this problem. We were stuck on looking at different examples, and one of my colleagues said, "Hey, you know, Maria, our president, who has a math background, is actually really good at constructing counter examples." So I called up my college president with little hope that she would have any time to think about this problem. I said, "Hey, Maria. We have this problem," and she said, "Oh, that's great. I'm sick with the flu, and I'm not traveling this weekend, and giving me a problem would take my mind off the flu." And so, it was crazy. On Monday, she came back, and she had a solution to the problem. She came up with some interesting, she proved, for instance something really that we didn't surprise. We were surprised that the most regular examples actually aren't the best examples. And I'll just, maybe, finish by saying that.

Dr. Francis Su: Here's the tantalizing result that everything we try, actually, seems to be a third. And so, here's some further questions you might think about. What about the agreement proportion for other spaces? Are there probabilistic voting results, like when you say some fraction of people pairwise agree, what can you say about the approval proportion? What can you say about doing this in practice, like, estimating approval proportions just from taking statistical samples? Lots of good questions.

Dr. Francis Su: I'll just close, then, by showing some examples of high-dimensional spaces that people actually care about. I showed you a two-dimensional example before. Here's a two-dimensional view of the state. State is good, state is bad. You see communists in one color, Natzis in the other. Anarchists and this. And this is Ayn Rand, in the other... okay? Here's a three-dimensional example people care about. This is from an economist website, where you have political liberty and personal liberty and... personal, political, and economic liberty along different axis. And would you believe I actually found a 29-dimensional example that people actually care about. For people are wondering about where they sit in a spectrum that's 29-dimensional, and it's called eharmony. The point is to land in lots of people's approval sets. Or maybe just one. You just need one.

Dr. Francis Su: Anyways, I hope I've convinced you that math can model social sciences, and that social sciences can motivate interesting math. Thank you very much.

Speaker 1: Do you have any-