Honors Examination: Topology
Swarthmore College, 2014

Please answer the following questions. In each case, give as complete a proof of your statements as you can. You may of course use any standard theorem, but please give precise statements of theorems you appeal to.

1. Let X be a topological space and $A \subseteq X$. Recall that the *frontier* of A is the subspace of X given by $\text{bd}X = \overline{A} \cap \overline{X - A}$, where $\overline{A}$ indicates the closure of A.

(a) Give an example of a subset of $\mathbb{R}$ (with its usual topology) whose frontier is the closed interval $[-1, 1]$.

(b) Suppose A and C are subspaces of X and that C meets both A and $X - A$. Show that if C is connected then C also meets $\text{bd}A$.

(c) Suppose that X and Y are topological spaces and that $f : X \to Y$ is a continuous function. Let A be a subset of X, and show that $f(\text{bd}A) \subseteq \text{bd}f(A)$.

2. Armstrong gives the following definition: A *surface* is a topological space in which each point has a neighborhood homeomorphic to the plane, and for which any two distinct points possess disjoint neighborhoods.

(a) Give an example of a space satisfying the first condition but not the second.

(b) Give an example of a surface which cannot be embedded into any finite dimensional Euclidean space.

3. Let 8 denote the space obtained from the disjoint union of two circles by identifying a point from the first with a point from the second. Its fundamental group is the free group on two generators a, b, with generators corresponding to the two loops in 8.

(a) Describe the covering space of 8 corresponding to the subgroup generated by a. Your description might include a picture. As part of your answer, you should describe the covering map.

(b) Do the same for the minimal *normal* subgroup containing a.

(c) What is the group of deck (or covering) transformations of the covering space defined in **(b)**? Describe how it acts, in terms of your answer to **(b)**.

4. Let D^2 be the closed unit disk in the complex plane, and define the following equivalence relation: $x \sim y$ if and only if either $x = y$ or $|x| = 1$ and $x^3 = y^3$ as complex numbers. Let X be the quotient space with respect to this equivalence relation.

(a) Is X connected?

(b) Is X compact?

(c) Is X Hausdorff?

(d) Is X a surface?

(e) Compute $\pi_1(X, o)$, where o is the class of the complex number 0.

(f) Show that X is a simplicial complex by describing (e.g. with a picture) an explicit triangulation.

(g) What is the Euler characteristic of X?

(h) Suppose that $f : X \to X$ is a continuous map. Is there $a \in X$ such that $f(a) = a$?

(i) Compute the simplicial homology of X.