SWARTHMORE COLLEGE HONORS EXAM 2014
REAL ANALYSIS

Instructions: Do as many problems, or parts of problems, or special cases of problems, as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove.

Real analysis I

1. A sequence of non-negative real numbers $a_1, a_2, \ldots$ is called *subadditive* if $a_{m+n} \le a_m + a_n$ for all $m, n \ge 1$. Show that for any subadditive sequence, $\lim_{n\to\infty} \frac{a_n}{n}$ exists and equals $\inf_{n\to\infty} \frac{a_n}{n}$.

2. Let (X, d) be a metric space, and let $\mathcal{K}(X)$ be the space of all nonempty compact subsets of X. We define the Hausdorff metric d_H on $\mathcal{K}(X)$ as follows: for A, $B \in \mathcal{K}(X)$, $d_H(A, B)$ is the smallest ε such that for every point a in A, there exists a point b in B with $d(a, b) \le \varepsilon$, and for every point b in B, there exists a point a in A with $d(a, b) \le \varepsilon$.

(a) Let $\mathcal{S}$ be the set of closed intervals in $\mathbb{R}$, that is, the set $\{[x, y] : x \le y\}$. Is $\mathcal{S}$ open in $\mathcal{K}(\mathbb{R})$ with the Hausdorff metric? Closed? Neither?

(b) Given a set Y in the space X, its *boundary*, $\operatorname{bd} Y$, is the set $\operatorname{bd} Y = \operatorname{cl}(Y) \cap \operatorname{cl}(X \backslash Y)$. Show that the map $\partial : \mathcal{K}(\mathbb{R}) \to \mathcal{K}(\mathbb{R})$ given by $\partial(Y) = \operatorname{bd} Y$ is well defined, and determine whether it is continuous under the Hausdorff metric.

(c) Let $\{f_n : [0, 1] \to \mathbb{R} : n = 1, 2, \ldots\}$ be a sequence of continuous functions. Prove or disprove: The functions $\{f_n\}$ converge to a function f uniformly on $[0, 1]$ if and only if the corresponding graphs, $\{(x, f_n(x)) \in \mathbb{R}^2 : x \in [0, 1]\}$, converge to the graph of f in $\mathcal{K}(\mathbb{R}^2)$ with the Hausdorff metric.

3. Recall the Intermediate Value Theorem:

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and y any number between $f(a)$ and $f(b)$ inclusive. Then there exists a point $c \in [a, b]$ with $f(c) = y$.

(a) Prove the Intermediate Value Theorem.

(b) Prove or disprove the following fixed point theorem:

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function, and x_1 and x_2 distinct points such that $g(x_1) = x_2$ and $g(x_2) = x_1$. Then there exists a fixed point x (that is, a point x such that $g(x) = x$).

4. (a) Prove the following attracting fixed point theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function, and let x_0 be a point such that $f(x_0) = x_0$ and $|f'(x_0)| < 1$. Then x_0 is attracting, that is, there is an interval I containing x_0 in its interior such that $f(I) \subset I$ and the sequence $x, f(x), f(f(x)), \ldots$ converges to x_0 for all x in I.*

(b) Show that the sequence defined by $s_1 = 1$ and $s_{n+1} = \frac{1}{2}\left(s_n + \frac{2}{s_n}\right)$ for $n = 1, 2, \ldots$ converges to $\sqrt{2}$. (This is the Babylonian method for computing square roots.)

5. Define a sequence of functions $f_1, f_2, \ldots : [0, 1] \to \mathbb{R}$ by $f_n(x) = \sqrt{n}x^n(1 - x)$. Discuss the convergence of $\{f_n\}$, $\{f_n'\}$, and $\{\int_0^1 f_n(x)\,dx\}$ as $n \to \infty$.

6. Prove that for every $n \times n$ matrix A sufficiently near the identity matrix, there is a square-root matrix B (i.e., a solution to $B^2 = A$). Show that the solution is unique if B must also be sufficiently near the identity matrix.

7. Let $T \subset \mathbb{R}^3$ be the torus $(2 - \sqrt{x^2+y^2})^2 + z^2 = 1$, and let ω be the 2-form, defined on $\mathbb{R}^3 \setminus \{\vec{0}\}$, given by

$$\omega = \frac{x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}.$$

(a) Show that ω is closed.
(b) Compute $\int_T \omega$.
(c) Compute $\int_{S^2} \omega$, where S^2 is the unit sphere in $\mathbb{R}^3$.

8. For what values of c will the set $\{(x,y,z) : x^3 + y^3 + z^3 - 2xyz = c\}$ be a 2-manifold?

9. (a) Compute $\iiint_{\mathbb{R}^3} f(2x, 3y, 4z)\,dx\,dy\,dz$, given that $\iiint_{\mathbb{R}^3} f(x,y,z)\,dx\,dy\,dz = 1$.
(b) Define $S \subset \mathbb{R}^2$ to be the set $\{(x,y) : -1 \le x \le 1,\ 0 \le y \le 2\sqrt{1-|x|}\}$. Compute

$$\iint_S \frac{\sqrt{2}}{2}\sqrt{\sqrt{x^2+y^2}+x}\,dx\,dy.$$

Hint: The function $g(u,v) = (u^2 - v^2, 2uv)$ maps the unit square $\{(u,v) : 0 \le u \le 1,\ 0 \le v \le 1\}$ one-to-one onto S.