SWARTHMORE COLLEGE HONORS EXAM 2015
REAL ANALYSIS

Instructions: Do as many problems, or parts of problems, or special cases of problems, as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove.

1. (a) Prove the following obscure theorem:

If f is a continuous function on the closed interval $[a, b]$, then $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$ *for* $x \in [a, b]$.

(b) What can you say if f not continuous?

2. Define a sequence of functions $f_1, f_2, \ldots : [-1, 1] \to \mathbb{R}$ by $f_n(x) = \sqrt{\dfrac{1}{n} + x^2}$. Discuss the convergence of $\{f_n\}$, $\{f_n'\}$, and $\{\int_0^1 f_n(x)\,dx\}$ as $n \to \infty$. If the convergence isn't "good," discuss what additional hypotheses would guarantee better convergence.

3. Let I be the unit interval $[0, 1]$, and let $f : I \to I$ be a function from I to itself such that for all $x, y \in I$, the inequality $d(f(x), f(y)) < d(x, y)$ holds.

(a) Show that there is a unique point $x_0 \in I$ such that $f(x_0) = x_0$.

(b) Prove or disprove: There exists a constant $c < 1$ such that $d(f(x), f(y)) \leq c \cdot d(x, y)$ for all x and y in I.

4. Let K be a compact subset of $\mathbb{R}^n$, and let $\mathcal{U}$ be a collection of open balls whose union contains K. Show that there exists a positive real number λ such that every subset of K of diameter less than λ is contained in some element of $\mathcal{U}$.

5. The Cantor set C is defined as follows: Let C_1 be the unit interval $[0, 1]$. Construct the set C_2 by removing the open middle third of C_1, that is, $C_2 = [0, 1/3] \cup [2/3, 1]$. Now define C_3 by removing the open middle third of each sub-interval making up C_2, that is $C_3 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. In general, construct C_n by removing the open middle third of each sub-interval making up C_{n-1}. The Cantor set C is the intersection of all the C_n's:

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Define the function $f : [0, 1] \to \mathbb{R}$ by $f(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{if } x \notin C. \end{cases}$

Is f Riemann integrable? If so, compute $\displaystyle\int_0^1 f(x)\,dx$.

6. For what values of c will the solution set of the equations

$$x^3 + y^3 + z^3 = c, \;\; z = xy$$

in $\mathbb{R}^3$ be a manifold?

7. Let $C = (0,1)^2$. Let $\alpha : C \to \mathbb{R}^3$ be given by the equation

$$\alpha(s,t) = (s+t, s^2, t^2).$$

Let M be the image $\alpha(C)$. Evaluate $\displaystyle\int_M dx \wedge dy + y\, dx \wedge dz.$

8. Let M be a $2n$-dimensional manifold. A 2-form ω defined on M is *symplectic* if it is closed and the $2n$-form $\omega \wedge \omega \wedge \ldots \wedge \omega$ (the product of ω with itself n times) is a volume form on M (that is, a nowhere-vanishing $2n$-form).

(a) Let S^{2n} be the unit sphere in $\mathbb{R}^{2n+1}$. Show that there are no exact symplectic forms on S^{2n}.

(b) Show that

$$\omega = \frac{x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$

is a symplectic form on S^2.

9. The *general linear group* $GL(n,\mathbb{R})$ is the group of $n \times n$ real invertible matrices. A group is a *Lie group* if it is also a manifold, and the operations of multiplication and taking inverses are differentiable. Show that $GL(n,\mathbb{R})$, considered as a subset of $\mathbb{R}^{n^2}$, is a Lie group. (You may assume that it's a group; you don't need to prove it.)