SWARTHMORE COLLEGE HONORS EXAM 2014
COMPLEX ANALYSIS

Instructions: Do as many problems, or parts of problems, or special cases of problems, as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove.

Real analysis I

1. A sequence of non-negative real numbers a_1, $a_2, \ldots$ is called *subadditive* if $a_{m+n} \leq a_m + a_n$ for all $m, n \geq 1$. Show that for any subadditive sequence, $\lim_{n\to\infty} \frac{a_n}{n}$ exists and equals $\inf_{n\to\infty} \frac{a_n}{n}$.

2. Let (X, d) be a metric space, and let $\mathcal{K}(X)$ be the space of all nonempty compact subsets of X. We define the Hausdorff metric d_H on $\mathcal{K}(X)$ as follows: for A, $B \in \mathcal{K}(X)$, $d_H(A, B)$ is the smallest ε such that for every point a in A, there exists a point b in B with $d(a, b) \leq \varepsilon$, and for every point b in B, there exists a point a in A with $d(a, b) \leq \varepsilon$.

(a) Let $\mathcal{S}$ be the set of closed intervals in $\mathbb{R}$, that is, the set $\{[x, y] : x \leq y\}$. Is $\mathcal{S}$ open in $\mathcal{K}(\mathbb{R})$ with the Hausdorff metric? Closed? Neither?

(b) Given a set Y in the space X, its *boundary*, $\operatorname{bd} Y$, is the set $\operatorname{bd} Y = \operatorname{cl}(Y) \cap \operatorname{cl}(X \backslash Y)$. Show that the map $\partial : \mathcal{K}(\mathbb{R}) \to \mathcal{K}(\mathbb{R})$ given by $\partial(Y) = \operatorname{bd} Y$ is well defined, and determine whether it is continuous under the Hausdorff metric.

(c) Let $\{f_n : [0, 1] \to \mathbb{R} : n = 1, 2, \ldots\}$ be a sequence of continuous functions. Prove or disprove: The functions $\{f_n\}$ converge to a function f uniformly on $[0, 1]$ if and only if the corresponding graphs, $\{(x, f_n(x)) \in \mathbb{R}^2 : x \in [0, 1]\}$, converge to the graph of f in $\mathcal{K}(\mathbb{R}^2)$ with the Hausdorff metric.

3. Recall the Intermediate Value Theorem:

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and y any number between $f(a)$ and $f(b)$ inclusive. Then there exists a point $c \in [a, b]$ with $f(c) = y$.

(a) Prove the Intermediate Value Theorem.

(b) Prove or disprove the following fixed point theorem:

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function, and x_1 and x_2 distinct points such that $g(x_1) = x_2$ and $g(x_2) = x_1$. Then there exists a fixed point x (that is, a point x such that $g(x) = x$).

4. (a) Prove the following attracting fixed point theorem. *Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function, and let x_0 be a point such that $f(x_0) = x_0$ and $|f'(x_0)| < 1$. Then x_0 is attracting, that is, there is an interval I containing x_0 in its interior such that $f(I) \subset I$ and the sequence $x, f(x), f(f(x)), \ldots$ converges to x_0 for all x in I.*

(b) Show that the sequence defined by $s_1 = 1$ and $s_{n+1} = \frac{1}{2}\left(s_n + \frac{2}{s_n}\right)$ for $n = 1, 2, \ldots$ converges to $\sqrt{2}$. (This is the Babylonian method for computing square roots.)

5. Define a sequence of functions f_1, $f_2, \ldots : [0, 1] \to \mathbb{R}$ by $f_n(x) = \sqrt{n}x^n(1 - x)$. Discuss the convergence of $\{f_n\}$, $\{f_n'\}$, and $\{\int_0^1 f_n(x)\, dx\}$ as $n \to \infty$.

Complex analysis

6. Recall the Fundamental Theorem of Algebra:

Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, with $a_i \in \mathbb{C}$ for all i, a_n nonzero, and $n \geq 1$. Then there exists a point $z_0 \in \mathbb{C}$ such that $p(z_0) = 0$.

(a) State Liouville's theorem and use it to prove the Fundamental Theorem of Algebra.
(b) State Rouche's theorem and use it to give another proof of the Fundamental Theorem of Algebra.

7. Recall the Schwarz Lemma:

Let f be analytic on the open unit disk A in $\mathbb{C}$, with $f(0) = 0$ and $|f(z)| \leq 1$ for each z in A. Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for each z in A. If $|f'(0)| = 1$ or if there is a point z_0 other than 0 in A with $|f(z_0)| = |z_0|$, then there is a constant c with $|c| = 1$ and $f(z) = cz$ for all z in A.

(a) Prove the Schwarz Lemma. Hint: Consider $f(z)/z$ and use the Maximum Modulus Principle.
(b) Prove or disprove the following theorem:

Let A be the open unit disk in $\mathbb{C}$, and let $f : A \to A$ be analytic. If $f(0) = 0$ and $f(1/2) = 1/2$, then f is the identity on A, that is, $f(z) = z$ for each z in A.

8. (a) Compute the integral $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 4}\, dx$.
(b) How many zeros, counting multiplicities, does the polynomial $z^7 - 2z^2 + 4$ have in the first quadrant?

9. We can think of a point $z = x + iy$ in $\mathbb{C}$ as corresponding to a point (x, y) in $\mathbb{R}^2$. Similarly, we can think of a function $f(z) = u(z) + iv(z)$ as corresponding to a vector field V_f on $\mathbb{R}^2$, given by $V_f(x, y) = (u(x + iy), v(x + iy))$. Assume that u and v have continuous partial derivatives with respect to x and y. Show that $f(z)$ is analytic on $\mathbb{C}$ if and only if the vector field $V_{\overline{f}}$ corresponding to $\overline{f(z)}$ is conservative and fluxless on $\mathbb{R}^2$. (Recall that a vector field V is conservative if the net work done by V around any smooth closed curve is 0; it's fluxless if the net flux of V through any smooth closed curve is 0.)