Swarthmore College
Department of Mathematics and Statistics
Honors Examination: Algebra

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Instructions: This exam consists of ten problems. Please try to do *six* of them as thoroughly as possible. Once you have done your best on those, make a second pass through the exam and do as many parts of the remaining problems as you can.

General hints and advice: If you get stuck, work out some examples or special cases. If it makes sense, formulate and solve an easier version of the problem. In general, I am interested in learning where your thoughts are going, even if you do not answer the question completely. When there are multiple parts to a problem, you may do them in any order. Please justify your reasoning as fully as possible.

1. Let $O(n)$ denote the orthogonal group (that is, the set of all real $n \times n$ matrices A such that $AA^T = I$), and let $SO(n)$ denote the subgroup consisting of matrices with determinant 1.
   (a) Prove that, for all n, $SO(n)$ is a normal subgroup of $O(n)$ of index 2.
   (b) For which n (if any) is $O(n)$ isomorphic to $SO(n) \times \{\pm 1\}$? Explain your answer carefully.
2. Let X denote the set of 2×2 complex matrices, and let $G = GL_2(\mathbb{C})$.
   (a) If G acts on X by left-multiplication, how does X decompose into orbits?
   (b) If G acts on X by conjugation, how does X decompose into orbits?
   (c) For each of the orbits in (a) and (b), pick a representative element and describe its stabilizer.
3. Let $\mathbb{F}_q$ denote the field with q elements, and let $SL_n(\mathbb{F}_q)$ denote the group of $n \times n$ matrices over $\mathbb{F}_q$ with determinant 1.
   (a) Show that $|SL_2(\mathbb{F}_3)| = 24$.
   (b) Is $SL_2(\mathbb{F}_3)$ isomorphic to the symmetric group S_4? Explain your answer carefully.
   (c) Let G be the group of rotations of a cube. Is G isomorphic to S_4? Explain your answer carefully.
4. Let $M(2)$ be the group of isometries of $\mathbb{R}^2$. Consider the subgroup $O(2)$ consisting of rotations and reflections fixing the origin, and the subgroup $T(2)$ consisting of translations.
   (a) Determine whether $O(2)$ and $T(2)$ (or perhaps both) are normal subgroups of $M(2)$.
   (b) Suppose that H is a subgroup of $M(2)$ containing rotations about two distinct points. Prove that H contains a nontrivial translation. *Hint: one approach (among several) is to look at commutators.*

5. Let G be a finite group.

   (a) Define the regular representation of G over the complex numbers. What is its character χ_R?

   (b) How does χ_R decompose into irreducible characters?

   (c) Show that if Ψ is a character of G and $\Psi(g) = 0$ for all $g \neq 1$ in G, then Ψ is an integral multiple of χ_R.

6. A famous theorem states that every finite abelian group G is isomorphic to a direct product of cyclic groups of prime power order, and that this representation is unique up to permutation of the factors. Prove the *second* part of this statement, i.e., uniqueness of the representation.

7. Let $R = \mathbb{Z}[x]$, the ring of polynomials in x with integer coefficients, and let $I = (3, x)$ be the ideal generated by 3 and x.

   (a) Is I principal? Is it prime? Is it maximal? Explain your answers.

   (b) Answer the questions in (a) for $J = (x^2 + 1)$, the ideal generated by $x^2 + 1$.

8. Let G be a finite, nonabelian simple group, and consider representations of G over the complex numbers.

   (a) Show that there cannot exist more than one linear (i.e., degree 1) character.

   (b) Show that there cannot exist any irreducible characters of degree 2.

9. Let G be a finite group, and let p be the smallest prime dividing $|G|$. Prove that any subgroup $H \subseteq G$ of index p is normal in G.

10. Let $\omega = e^{2\pi i/n}$ and let $F_\omega = \mathbb{Q}(\omega)$.

    (a) What is $[F_\omega : \mathbb{Q}]$?

    (b) Let $G = \mathrm{Gal}(F_\omega, \mathbb{Q})$ be the Galois group of F_ω over $\mathbb{Q}$. Describe G and compute its order.

    (c) List all of the subfields of F_ω when $n = 8$, and make a diagram showing their containment relationships.

    (d) *(If you have time)* For which n is G a cyclic group? Discuss, giving arguments and/or examples to support your answer.