Instructions. Choose at least 5 of the following problems, including at least 2 from Part A and at least 2 from Part B. If you cannot completely solve a problem, include whatever progress you can make (even if it means sketching an argument without verifying each intermediate claim). You are invited to work on as many problems as you desire.

If you invoke a theorem, state clearly what the theorem says and explain why it applies. Unless a problem explicitly asks for a proof of a standard result, you may use results from your courses.

Part A

1. Given a set X, a topological space Y, and a function $f : X \to Y$, consider the set

$$\mathcal{T} = \{f^{-1}(U) : U \subset Y \text{ is open}\}.$$

(a) Show that $\mathcal{T}$ is a topology on X.

(b) Show that $\mathcal{T}$ is the coarsest topology on X that makes f continuous. (That is, if $\mathcal{T}'$ is any other topology on X that makes f continuous, then $\mathcal{T}'$ contains $\mathcal{T}$.)

(c) Show that if $(X, \mathcal{T})$ is Hausdorff, then f is injective.

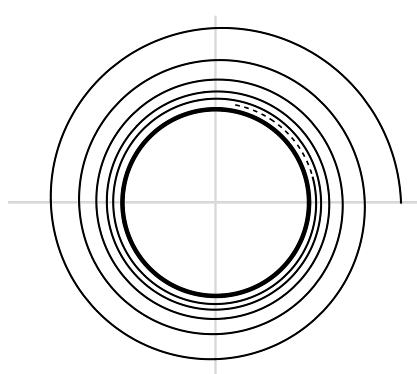
2. Let X be a compact metric space, and let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ be a nested sequence of nonempty closed subsets whose diameters converge to zero. (Recall that the *diameter* of a set A is $\operatorname{diam}(A) = \sup\{d(x,y) : x, y \in A\}$.) Show that $\cap_{n=1}^{\infty} A_n$ consists of exactly one point.

3. **(a)** Show that a continuous map that is surjective and open is a quotient map.

(b) Let $X = [0,1]$ and $A = \{0\} \cup \left\{ {}^{1}\!/_{n} : n \in \mathbb{Z}_{>0} \right\}$. Show that the quotient map $q : X \to X/A$ is closed but not open. *(It may help to draw a picture of X/A.)*

4. **(a)** Prove that if A is a connected subspace of a topological space X, then its closure $\bar{A}$ is also connected.

(b) Let $S \subset \mathbb{R}^2$ be the subset depicted below, which is the union of the unit circle and the curve $\gamma(t) = \big((1 + e^{-t})\cos(t), (1 + e^{-t})\sin(t)\big)$ for $t \in [0, \infty)$, which spirals towards the unit circle but never reaches it. Show that S is connected but not path connected. *(While a full proof is ideal, a convincing sketch is acceptable.)*



Part B

5. Let X and Y be spaces that are compact, Hausdorff, and path-connected. Given a continuous map $f : X \to Y$, consider the induced homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$.

Determine whether each statement below is true or false, justifying each answer with a proof or counterexample. *Do not forget the hypotheses placed on X and Y!*

(a) If f is injective, then f_* is injective.

(b) If f is surjective, then f_* is surjective.

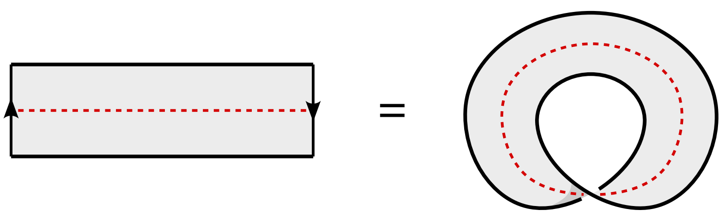
(c) If f is bijective, then f_* is an isomorphism.

6. Let M be the Möbius band, as shown below; its core circle is depicted as a dashed curve.

(a) Show that M deformation retracts onto its core circle. Deduce that $\pi_1(M) \cong \mathbb{Z}$.

(b) Let ∂M denote the boundary of M, indicated by the thick black curve in the right-hand side of the figure below. Determine $\pi_1(\partial M)$ and calculate the inclusion-induced homomorphism $i_* : \pi_1(\partial M) \to \pi_1(M)$.

(c) Show that ∂M is not a retract of M. (Recall that a subspace A of a topological space X is a *retract* if there is a continuous map $r : X \to A$ that is a left inverse of the inclusion map $i : A \hookrightarrow X$, i.e., $r \circ i = \mathrm{id}_A$.)



7. Let K be the Klein bottle, as shown below.

(a) Calculate $\pi_1(K)$.

(b) Show that there is a degree-2 covering map $p : T^2 \to K$ from the torus to the Klein bottle and identify the subgroup $p_*(\pi_1(T^2))$ inside $\pi_1(K)$.

(c) Show that there is no covering map $q : K \to T^2$.