Instructions

Choose at least 5 of the following problems, including at least one from part A and at least one from part B. If you use any theorems be sure to explain precisely what they say and why they apply.

Part A

1. Suppose that X, Y are topological spaces and $p : X \to Y$ is a quotient map.
   (a) Show that if the set $p^{-1}(y)$ is connected for all $y \in Y$ and Y is connected, then X must also be connected.
   (b) Is the statement you proved in part (a) true if we weaken the assumption so that p is a continuous surjective map, but not necessarily a quotient map?

2. Consider the following theorem:

   Theorem 1. *Let X, Y be topological spaces with X compact and Y Hausdorff. Then every continuous bijective map $f : X \to Y$ is a homeomorphism.*

   (a) Define each of the underlined terms in the theorem.
   (b) Give a proof of this theorem. You may use any fact or theorem you have learned (except Theorem 1 itself) but be sure to explain exactly what the result you are citing says and why it applies here.

3. Consider two topological spaces: the Hawaiian earring is the countable union of circles of radius $\frac{1}{n}$ and center $(\frac{1}{n}, 0)$ as a subset of $\mathbb{R}^2$, with the subspace topology.

$$\mathbb{H} = \bigcup_{n=1}^{\infty} \left\{ (x, y) \in \mathbb{R}^2 : \left(x - \frac{1}{n}\right)^2 + y^2 = \left(\frac{1}{n}\right)^2 \right\}$$

   The infinite rose is the wedge sum of countably many circles joined at a point. The topology on this space comes from the wedge construction.

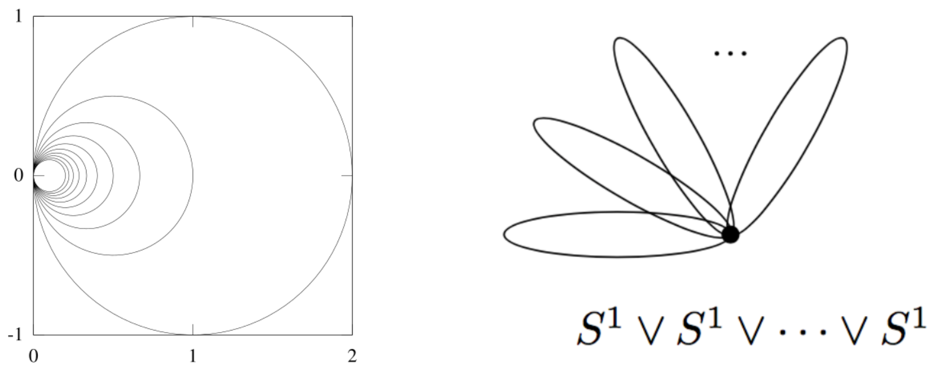


Figure 1: To the left are the 10 largest circles in the Hawaiian earring, to the right is wedge of n circles. Imagine infinitely many circles in both spaces.

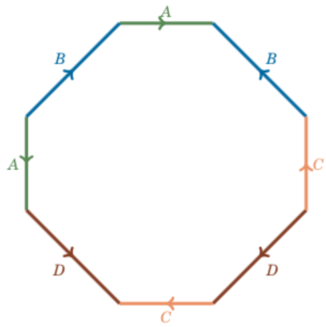
   (a) Show, via the definition of compactness, that the infinite rose is not a compact space.
   (b) Show that the Hawaiian earring is a closed set in $\mathbb{R}^2$. How does this imply that the Hawaiian earring is compact?

Part B

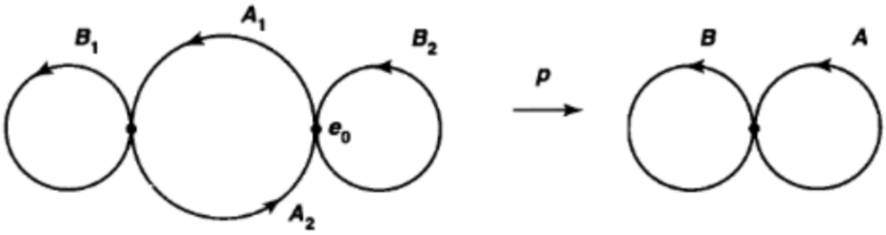
4. Identify the fundamental group of each of the following spaces. You may use the facts that the point and the sphere are simply connected, that the circle has fundamental group $\mathbb{Z}$, and that the torus T has fundamental group $\mathbb{Z} \times \mathbb{Z}$ but otherwise you should justify your answer.
   (a) The solid torus $D^2 \times S^1$.
   (b) The union of the unit circle and the x-axis, as a subset of $\mathbb{R}^2$.
   (c) A sphere with three punctures.
   (d) $T\#S^2$, the connected sum of the torus and the sphere.

5. Give examples of pairs of topological spaces with the following properties. Be sure to briefly justify why your examples satisfy the desired properties.
   (a) A pair of topological spaces X, Y such that X, Y are homotopy equivalent but not homeomorphic.
   (b) A pair of topological spaces X, Y such that $\pi_1(X) = \pi_1(Y)$ but X is not a deformation retract of Y and Y is not a deformation retract of X.

6. Here is a problem about the Classification of Surfaces:
   (a) State the Classification of Surfaces theorem.
   (b) Consider the regular octagon pictured below. If you identify the sides using a quotient map as indicated by the labels in the diagram, what familiar surface do you obtain. How does this surface fit into the Classification of the Surfaces theorem?



7. Consider $S^1 \vee S^1$ the wedge of two circles, with fundamental group with $F_2 = \langle a, b \rangle$, the free group on 2 generators a and b.
   (a) What is the universal cover of $S^1 \vee S^1$. Draw a picture and explain why this space is simply connected.
   (b) Consider the intermediate covering space pictured below, where each A_i maps to the circle A and each B_i maps to the circle B. What is the fundamental group of this covering space and how can you see it as a subgroup of F_2?



(c) Find an example of a covering space of $S^1 \vee S^1$ with the fundamental group F_∞, the free group on countably infinite generators. Draw a picture of this covering space and explain why its fundamental group has infinitely many generators.

8. The real projective plane $\mathbb{R}P^2$ is a non-orientable surface that can be viewed as the quotient of the 2-sphere under the map $q : S^2 \to \mathbb{R}P^2$ that identifies each point on the sphere with its antipodal point. Explain how you might use covering space theory to prove that $\mathbb{R}P^2$ is $\mathbb{Z}/2\mathbb{Z}$ the group of order 2. You don't have prove every lemma, just give an outline of the main results you would use and big ideas.

9. Here is a question about topological groups.
   (a) Give the definition of a topological group.
   (b) Show that $\mathcal{C} = \{z \in \mathbb{C} : |z| = 1\}$ is an example of a topological group, where the group operation is multiplication in $\mathbb{C}$. Identify a familiar topological space that is homeomorphic to $\mathcal{C}$.
   (c) Suppose that $(G, \cdot)$ is an arbitrary topological group with α a fixed element of G. Define a map $\phi_\alpha : G \to G$ given by $\phi_\alpha(g) = \alpha \cdot g$. Show that ϕ_α is a homeomorphism.