Swarthmore College
Department of Mathematics and Statistics
Honors Examination in Topology 2021

Do as many of the following 12 problems as thoroughly as you can in the time you have. Aim to include at least 2 problems form Section I and 2 problems from Section II.

Section I: Point Set Topology

(1) On the set $\mathbb{R}$, consider the collection $\mathcal{T}_0$ consisting of the empty set and all subsets of $\mathbb{R}$ that contain $0 \in \mathbb{R}$:

$$\mathcal{T}_0 = \{U \subset \mathbb{R} : U = \emptyset \text{ or } 0 \in U\}.$$

(a) Prove that $\mathcal{T}_0$ is a topology on $\mathbb{R}$.

(b) Is $(\mathbb{R}, \mathcal{T}_0)$ a Hausdorff space?

(c) Recall that in a topological space $(X, \mathcal{T})$, a sequence of points $(x_n)_{n \in \mathbb{Z}^+}$, $x_n \in X$, converges to $p \in X$ if for every neighborhood U of p, there exists $N \in \mathbb{Z}^+$ such that $x_n \in U$ when $n \geq N$.

(i) In $(\mathbb{R}, \mathcal{T}_0)$, does the sequence $(0, 0, 0, \dots)$ converge to 5? Explain your reasoning.

(ii) In $(\mathbb{R}, \mathcal{T}_0)$, does the sequence $(1, 1/2, 1/3, \dots)$ converge to 0? Explain your reasoning.

(2) Let $f, g : X \to Y$ be continuous maps. Suppose $D \subset X$ is a set such that that for every $x \in X$, every neighborhood U of x must intersect D.

(a) Show that if Y is Hausdorff and $f(x) = g(x)$ for all $x \in D$, then $f(x) = g(x)$ for all $x \in X$.

(b) Given a example showing that it is necessary for Y to be Hausdorff in order the equality of f and g on D to imply the equality of f and g on all of X.

(3) (a) Prove that every compact subspace of a Hausdorff space is closed.

(b) Give an example of a topological space X and a compact subspace that is not closed.

(4) (a) Prove that the continuous image of a path connected space is path connected.

(b) Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$:

$$S^n = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_1^2 + \cdots + x_{n+1}^2 = 1\right\}.$$

Prove that for all $n \geq 1$, S^n is path connected.

(5) (a) Suppose the X is connected and $f : X \to \mathbb{R}$ is a continuous map. Prove that if $p, q \in f(X)$ and $p \leq r \leq q$, then $r \in f(X)$.

(b) Suppose $f : S^2 \to \mathbb{R}$ is continuous. Prove that there exists $c \in S^2$ such that $f(c) = f(-c)$.

(6) An *arithmetic progression* in $\mathbb{Z}$ is a set $A_{a,b} = \{a + nb \mid n \in \mathbb{Z}\}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$.

(a) Prove that the collection of arithmetic progressions

$$\mathcal{A} = \{A_{a,b} : a, b \in \mathbb{Z}, b \neq 0\}$$

is a basis for a topology on $\mathbb{Z}$. This is called the *arithmetic progression topology.*

(b) Prove that $\mathbb{Z}$ with the arithmetic progression topology is regular. (Recall that this means that with respect to this topology on $\mathbb{Z}$, $\{z\}$ is closed, for all $z \in \mathbb{Z}$, and if B is a closed set in $\mathbb{Z}$ such that $z \notin B$, then there exist disjoint open sets containing z and B.)

(c) Is $\mathbb{Z}$ with the arithmetic progression topology metrizable? Justify your answer.

Section II. Algebraic Topology

(7) Suppose that $A \subset \mathbb{R}^n$ is star convex, meaning that for some point $a_0 \in A$, all the line segments joining a_0 to other points of A lie in A.

(a) Sketch a star convex set that is not convex.

(b) Prove that if A is star convex, then A is simply connected.

(8) Recall that a topological space X is *contractible* if the identity map $i_X : X \to X$ is homotopic to a constant map. Let $[X, Y]$ denote the set of homotopy classes of maps of X into Y.

(a) Show that if Y is contractible, then for any X, the set $[X, Y]$ has a single element.

(b) Show that if X is contractible and Y is path connected, then $[X, Y]$ has a single element.

(9) (a) Suppose $A \subset X$, and $r : X \to A$ is a retraction, namely r is a continuous map such that $r(a) = a$, for each $a \in A$. If $a_0 \in A$, show that the homomorphism induced by r,

$$r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$$

is surjective.

(b) Calculate the fundamental group of the following subsets of $\mathbb{R}^2$:

(i) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$;

(ii) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$;

(iii) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \geq 1\}$;

(iv) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \cup (\mathbb{R}^+ \times \{0\})$;

(v) $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \cup \{(x, 0) \in \mathbb{R}^2 \mid x \in [-1, 1]\}$.

(10) (a) Let $\mathbb{P}^2$ denote the projective plane. Explain why $\pi_1(\mathbb{P}^2, p_0)$ is a group of order 2.

(b) Suppose X is the topological space obtained by taking two copies of $\mathbb{P}^2$ and identifying a single point p in one copy with a single point q in the other copy.

(i) Determine the fundamental group of X.

(ii) Draw a picture of the universal covering space of X.

(11) (a) State the Classification Theorem for closed surfaces.

(b) Consider the following three surfaces: the sphere S^2, the torus T^2, and the projective plane $\mathbb{P}^2$. Explain how one can determine that no two of these surfaces are homeomorphic.

(c) Consider the surface X given as the connect sum of two tori and a projective plane:

$$X = T^2 \# T^2 \# \mathbb{P}^2.$$

Where does this surface lie on the classification you gave in (a)? Explain your reasoning.

(12) Let G be the"figure-8" graph, namely the space $S^1 \vee S^1$, which is the quotient of the disjoint union $S^1 \amalg S^1$ obtained by identifying a point x_0 of the first circle with a point y_0 of the second circle.

(a) Sketch the universal cover of G.

(b) Sketch two non-homeomorphic covering spaces of G where each of the covering spaces are connected and non-simply connected.