Swarthmore College
Department of Mathematics and Statistics
Honors Examinations in Topology 2019

Instructions: Do as many of the following problems as thoroughly as you can in the time you have. Include at least one problem from each of the four parts of the exam.

Part I: Point Set Topology

1) For each integer $n \geq 1$ consider the following subsets of $\mathbb{R}^2$ (with the euclidean topology):

$$W_n = \{(x, y) \in \mathbb{R}^2 \,|\, (x - n)^2 + y^2 = n^2\}\,;$$
$$X_n = \{(x, y) \in \mathbb{R}^2 \,|\, (nx - 1)^2 + n^2y^2 = 1\}\,;$$
$$Y_n = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + n^2y^2 = n^2\}\,;$$
$$Z_n = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + n^2y^2 = 1\}\,.$$

Group the subspaces $W = \bigcup_{n=1}^{\infty} W_n$, $X = \bigcup_{n=1}^{\infty} X_n$, $Y = \bigcup_{n=1}^{\infty} Y_n$, and $Z = \bigcup_{n=1}^{\infty} Z_n$ into homeomorphism classes.

2) Let X be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. For each $t \in \mathbb{R}$, let X_t be the subset of all $f \in X$ such that

$$\sup_{x \in \mathbb{R}} f(x) < t\,.$$

Let τ be the coarsest topology on X that contains X_t for each $t \in \mathbb{R}$.

a) Show that (X, τ) is not Hausdorff.

b) Show that every subset $K \subseteq (X, \tau)$ that contains the identity function is compact.

c) Let $Y \subseteq (X, \tau)$ be the subset of all constant functions. Is Y compact? Is it connected? Is it path-connected?

3) Let I be the interval $[0, 1]$ with the euclidean topology, let X be a compact Hausdorff topological space, and let $f : X \to X$ be a continuous bijection. Show that the quotient Y of $X \times I$ obtained by identifying $(x, 0)$ with $(f(x), 1)$ for each $x \in X$ is compact Hausdorff.

Part II: Homotopy

4) Let G be the group of all matrices of the form

$$\begin{bmatrix} 1 & l & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$$

for some $l, m, n \in \mathbb{Z}$. Let X be the quotient of $\mathbb{R}^3$ (with the euclidean topology) by the action of G defined by

$$\begin{bmatrix} 1 & l & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} (x, y, z) = (x + l, y + m + lz, z + n)\,.$$

a) Show that the projection $\mathbb{R}^3 \to X$ is a covering of infinite degree.

b) Show that X is path connected.

c) Calculate $\pi_1(X)$.

d) Show that X admits a degree n covering for each integer $n \geq 2$.

5) Let F be a finite subset of S^2, let $X = S^2 \setminus F$ (i.e. the 2-sphere with $|F|$ points removed) and let $Y = S^2/F$ (i.e. the sphere with $|F|$ points identified).

   a) Calculate $\pi_1(X)$.

   b) Calculate $\pi_1(Y)$.

   c) Are X and Y homeomorphic?

6) Consider the subset of $\mathbb{R}^3$ consisting of points $P_0 = (0,0,0)$, $P_1 = (1,0,0)$, $P_2 = (0,1,0)$ and $P_3 = (0,0,1)$, let

$$X = \bigcup_{0 \leq i < j \leq 3} \{tP_i + (1-t)P_j \mid t \in [0,1]\}.$$

   a) Show that X and $\mathbb{R}^3 \setminus X$ are both connected.

   b) Calculate $\pi_1(X)$.

   c) Calculate $\pi_1(\mathbb{R}^3 \setminus X)$.

Part III: Homology

7) Let X be a simplicial complex homeomorphic to a surface of genus g.

   a) Show that X has at least $\frac{7+\sqrt{1+48g}}{2}$ vertices.

   b) Find a triangulation of X with the minimum possible number of edges when $g = 1$.

   c) Assume that every vertex belongs to the same number d of edges. Find an upper bound for d that is sharp when $g = 1$.

8) Let X be a topological space.

   a) Given an abelian topological group G, show that pointwise multiplication induces a structure of abelian group on the set $[X, G]$ of equivalence classes of continuous maps $f : X \to G$ up to homotopy.

   b) Given closed subspaces U, V of X, show that there is an exact sequence

$$0 \to [X,\mathbb{Z}] \to [U,\mathbb{Z}] \oplus [V,\mathbb{Z}] \to [W,\mathbb{Z}] \to [X,S^1] \to [U,S^1] \oplus [V,S^1] \to [W,S^1]$$

   of abelian groups, where S^1 is identified with the multiplicative group of unit complex numbers, $\mathbb{Z}$ has the discrete topology, and $W = U \cap V$.

   c) Use part b) to calculate $[S^2, S^1]$.

9) Let $\tau_n : S^n \to S^n$ be the antipodal map defined by $\tau_n(x) = -x$ for all $x \in S^n$ (with respect to the standard embedding of S^n in $\mathbb{R}^{n+1}$). Let X be simply connected and let $f : X \to X$ be an involution (i.e. $f^2 = \mathrm{Id}_X$). Construct a continuous function $g : S^2 \to X$ such that $g \circ \tau_2 = f \circ g$ and show that there is no continuous function $h : X \to S^1$ such that $h \circ f = \tau_1 \circ h$.

Part IV: Essays

10) Write a short essay on Fixed Point Theorems that you know, including a discussion of topological spaces that have the Fixed Point Property (every continuous self map has a fixed point). Include at least one proof.

11) Write a short essay on the triangulation and classification of topological surfaces. Include at least one proof.

12) Write a short essay on existence and uniqueness theorems for covering spaces. Include at least one proof.