Swarthmore College
Honors Examination in Topology, 2015

Instructions. Solve at least 6 problems, at least one from each section, and ideally at least two from each section. (There are three sections.)

(I) Point-set topology

1. (a) Let X be a metric space and let K be a compact subset of X. Prove that K is bounded with respect to the metric on X, and is also closed. (Just use the definitions here: no need to cite any theorems.)

   (b) Find a metric space in which not every closed bounded subspace is compact.

2. (a) Prove that any path connected space must be connected. (Similarly, just use the basic definitions.)

   (b) Find a connected space which has uncountably many path-connected components.

3. Consider $\{0, 1\}$ with the discrete topology.

   (a) Let X be the countably infinite product $\prod\{0, 1\}$ in the product topology. Prove, without using Tychonoff's theorem, that X is compact.

   (b) Let Y be the countably infinite product $\prod\{0, 1\}$ in the box topology. Prove that Y is not limit point compact. (Recall that a space Y is *limit point compact* if any infinite subset of Y has a limit point in Y.)

(II) π_1, homotopy

4. Given a finite abelian group G, construct a topological space X for which $\pi_1(X) \cong G$.

5. For which compact surfaces M is there a covering map $S^2 \to M$? (Consider surfaces both with and without boundary.)

6. A space X is *contractible* if the identity map on X is null-homotopic.

   (a) Prove that if X is contractible, then X is path-connected.

   (b) A subset X of $\mathbb{R}^n$ is called *star convex* if there is a point $x_0 \in X$ such that, for every $x \in X$, the line segment from x_0 to x lies in X. Prove that if X is star convex, then X is contractible.

   (c) Suppose that A and B are subsets of a space X with $X = A \cup B$, and suppose that A, B, and $A \cap B$ are contractible. Must X be contractible?

(III) Homology

7. Suppose that X is a simplicial complex with a subcomplex A. Suppose that both A and X are path-connected, and suppose that

$$H_1(X,A) \cong \mathbb{Z}/2\mathbb{Z}, \quad H_2(X,A) \cong 0.$$

   (a) Suppose that $H_1(A) \cong \mathbb{Z}/2\mathbb{Z}$. What can you say about $H_1(X)$?

   (b) Suppose that $H_1(A) \cong \mathbb{Z}/3\mathbb{Z}$. What can you say about $H_1(X)$?

   (c) Suppose that $H_1(A) \cong \mathbb{Z}$. What can you say about $H_1(X)$?

8. Let F be a field. Suppose that

$$0 \to C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \to 0$$

   is a chain complex in which each C_i is a finite-dimensional F-vector space and each d_i is a linear transformation. Let H_i be the homology of this chain complex in the ith position. Prove that

$$\sum_{i=0}^{n} (-1)^i \dim C_i = \sum_{i=0}^{n} (-1)^i \dim H_i.$$

   What topological conclusions can you draw from this?

9. Consider the diagram of abelian groups

$$\begin{array}{ccccccc} & & & & X & & \\ & & & & \big\downarrow{\scriptstyle f} & & \\ A & \xrightarrow{h} & B & \xrightarrow{i} & C & \xrightarrow{j} & D \end{array}$$

   where the bottom row $A \to B \to C \to D$ is exact. Show that i and f induce well-defined homomorphisms $\tilde{i} : B/\operatorname{im} h \to C$ and $\tilde{f} : \ker(j \circ f) \to B/\operatorname{im} h$ such that $\tilde{i} \circ \tilde{f}(x) = f(x)$ for all $x \in \ker(j \circ f)$.