Swarthmore College
Department of Mathematics and Statistics
Honors Examination in Topology 2012

Instructions: Do 6 of the following 10 problems as thoroughly as you can. Include at least one problem from each of the four parts of the exam. You may use without proof the basic theorems that you have learned, but be sure to state them carefully. You may also use a result from one problem in solving another, even if you didn't solve the first problem. If you have lots of time left over, feel free to solve other problems.

Notation

The following conventions for notation have been followed:
$\mathbb{Z}$ = the set (group, ring) of rational integers.
$\mathbb{Q}$ = the set (group, field) of rational numbers.
$\mathbb{R}$ = the set (group, field) of real numbers.
$\mathbb{C}$ = the set (group, field) of complex numbers.
S^{n-1} = the unit sphere in $\mathbb{R}^n$.

Point Set Topology

1. Consider the following collection of subsets of the natural numbers $\mathbb{N} = \{1, 2, 3, \ldots, \}$: We will say that a proper subset K is *closed* in $\mathbb{N}$ if $K = \{a_i\}$ and $\sum_i \dfrac{1}{a_i}$ is finite. We also take $\mathbb{N}$ to be closed. Prove that this collection determines a topology on $\mathbb{N}$ and prove that $\mathbb{N}$ is connected in this topology. Is it Hausdorff?

2. A function $f\colon (X, d) \to (Y, d')$ is uniformly continuous if for each $\epsilon > 0$, there is a $\delta > 0$ so that, for all x, y in X, $d(x, y) < \delta$ implies $d'(f(x), f(y)) < \epsilon$. Prove that if (X, d) is a compact metric space and $f\colon (X, d) \to (Y, d')$ is continuous, then f is uniformly continuous.

3. Consider the equivalence relation on $\mathbb{R}$ given by $x \sim y$ where $x \sim y$ if x and y are integers, and $x \sim y$ means $x = y$ if x or y is not an integer. Consider the quotient map $p\colon \mathbb{R} \to \mathbb{R}/\!\sim$. Prove that p is a closed mapping. Is $\mathbb{R}/\!\sim$ Hausdorff? Is $\mathbb{R}/\!\sim$ locally compact, that is, for any $[x] \in \mathbb{R}/\!\sim$ and open set U with $[x] \in U$, there is a compact set $K \subset \mathbb{R}/\!\sim$ with $[x] \in (\text{interior}\, K) \subset K \subset U$.

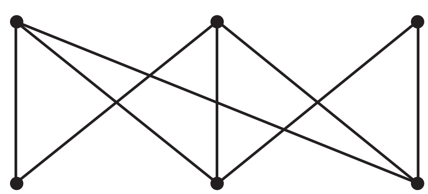
Fundamental Groups and Covering Spaces

4. A continuous closed curve $\gamma\colon [0,1] \to \mathbb{C} - \{0\}$, for which $\gamma(0) = \gamma(1) = 1 \in \mathbb{C}$ has *winding number* $wn(\gamma)$ defined by the function $\tilde{\gamma}(s) = \gamma(s)/|\gamma(s)| = e^{ig(s)}$ for a choice of continuous function $g\colon [0,1] \to \mathbb{R}$ and $wn(\gamma) = \dfrac{g(1)-g(0)}{2\pi}$. Prove that the inclusion $S^1 \to \mathbb{C} - \{0\}$ induces an isomorphism of fundamental groups, $\mathbb{Z} \cong \pi_1(S^1, 1) \cong \pi_1(\mathbb{C} - \{0\}, 1)$ and that the homotopy class of the curve $[\gamma] \in \pi_1(\mathbb{C} - \{0\}, 1)$ determines an integer, which is $\pm wn(\gamma)$.

5. A subset of the torus $S^1 \times S^1$ is a *punctured torus* $S^1 \times S^1 - \{p\}$, where p is some point in the torus. Show that the punctured torus has a figure eight ($S^1 \vee S^1$) as retract, and consider the homomorphism $i_*\colon \pi_1(S^1 \times S^1 - \{p\}, q) \to \pi_1(S^1 \times S^1, q)$ induced by the inclusion. Describe the homomorphism and justify your answer.

Surfaces

6. An embedding of a graph in the plane can be construed as an embedding into the sphere S^2 and the edges break the sphere up into faces, leading to an occasion to apply Euler's formula. Let $K_{3,3}$ denote the complete bipartite graph known as the houses-utilities graph. Prove that $K_{3,3}$ is not planar using Euler's formula. From the graph you can get the number of faces, and from the graph you can deduce the minimum number of edges for each face (a face corresponds to a circuit in the graph). Counting edges from the faces leads to a desired contradiction.



$K_{3,3}$

7. Let $M(S^1)$ denote the following set, $M(S^1) = \{\{u, v\} \subset S^1\}$, that is, the set of all subsets of S^1 of the form $\{u, v\}$. Notice that this does *not* preclude $u = v$. We can topologize $M(S^1)$ as a quotient of $S^1 \times S^1$. Show that $M(S^1)$ is homeomorphic to the Möbius band, with the boundary of this Möbius band identified with S^1.

8. On *Busy Bee World*, the entire surface of this spherical planet is covered by a hive that is one cell deep, and the cells take the shape of a hexagon (when all is well) or a pentagon (in a few cases). Prove that there are exactly 12 pentagons under these assumptions.

Algebraic Topology

9. Homology with coefficients can sometimes be useful to distinguish spaces. Compute the homology of $\mathbb{R}P^2$, the real projective plane, with coefficients in the field $\mathbb{Q}$. Use your calculation to prove that every mapping $f\colon \mathbb{R}P^2 \to \mathbb{R}P^2$ has a fixed point.

10. A *graded vector space* is a sequence of vector spaces over a field $\mathbb{F}$, $\{V_0, V_1, V_2, \ldots, V_n, \ldots\}$, and denoted $V_\bullet$. The *Euler characteristic for a graded vector space* is defined by $\chi(V_\bullet) = \sum_{i=0}^{N} (-1)^i \dim V_i$.

Suppose that $A_\bullet$, $B_\bullet$ and $C_\bullet$ are graded vector spaces for which A_n, B_n and C_n are finite dimensional for all n. Suppose further that there is an integer $N > 0$ with $A_{N+k} = \{0\}$, $B_{N+k} = \{0\}$, and $C_{N+k} = \{0\}$, for all $k > 0$. Finally, suppose that there is a sequence of linear transformations which is exact:

$$\begin{aligned} 0 \to A_N \to B_N \to C_N \to A_{N-1} \to \cdots \\ \to A_1 \to B_1 \to C_1 \to A_0 \to B_0 \to C_0 \to 0. \end{aligned}$$

(a) Show that $\chi(B_\bullet) = \chi(A_\bullet) + \chi(C_\bullet)$.

(b) The *Mayer-Vietoris sequence* is an exact sequence associated to a pair of subcomplexes A and B of a finite simplicial complex X, such that $X = A \cup B$. Suppose $H_k(\ ) = H_k(\ ;\mathbb{F})$ denotes homology with coefficients in the field $\mathbb{F}$. Then the following sequence is exact:

$$\begin{aligned} \cdots \to H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to H_k(X) \to H_{k-1}(A \cap B) \to \\ \cdots \to H_1(X) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(X) \to 0. \end{aligned}$$

Use (a) to prove that $\chi(A) + \chi(B) - \chi(A \cap B) = \chi(X) = \chi(A \cup B)$. (This identity resembles a basic property of a measure.)