Swarthmore 2011 Honors Exam in Topology

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Do 6 of the 10 problems below as completely as you can, including at least one from each of the three parts.

Point-set topology

(1) Prove that a topological space X is connected if and only if every continuous map from X to the rational numbers $\mathbb{Q}$ (topologized as a subspace of the real line) is constant.

(2) Recall that a topology on X is called T_1 if points are closed. Find a T_1 topology on the real line $\mathbb{R}$ in which every subset of $\mathbb{R}$ is compact.

(3) Let P^2 denote the real projective plane, a point of which is a line through the origin in $\mathbb{R}^3$.

(a) Describe P^2 as a quotient space of the sphere S^2 via a quotient map $p\colon S^2 \to P^2$.

(b) Define $f\colon S^2 \to \mathbb{R}^4$ for

$$f(x,y,z) = (x^2 - y^2, xy, xz, yz).$$

Prove that there is a continuous map $\overline{f}\colon P^2 \to \mathbb{R}^4$ such that $\overline{f} \circ p = f$.

(c) Prove that $\overline{f}$ is an embedding; that is, prove that P^2 is homeomorphic to the subspace $\overline{f}(P^2)$ of $\mathbb{R}^4$.

Fundamental group and surfaces

(4) Suppose X is a path connected, locally path connected topological space with $\pi_1(X, x_0) = \mathbb{Z}/2011\mathbb{Z}$. Prove that every map $X \to S^1$ is nulhomotopic.

(5) Consider the Klein bottle K and the torus T.
  (a) Show that there is a 2-fold covering map $T \to K$.
  (b) Show that there is a 3-fold covering map $K \to K$.
  (c) Generalize the preceding two parts from 2 and 3 to all positive integers. That is, for every positive integer n, construct a $2n$-fold covering map from T to K, and a $2n-1$-fold covering map from K to K.

(6) (a) Describe the fundamental group and the universal cover of the projective plane P^2.
  (b) Describe the fundamental group and universal cover of $P^2 \vee P^2$, the space consisting of two projective planes tied together at one point.

(7) Give a construction of a path connected space whose fundamental group is S_3, the group of permutations of a set with 3 elements. It might help to write down generators and relations for S_3.

Homology

(8) Find all the homology groups of the 3-skeleton of the 5-simplex. (The 3-skeleton is the subcomplex consisting of all the simplices of dimension at most 3).

(9) Giving two positive integers n and k, the **Moore space** $M^n(k)$ is the space $S^n \cup_f e^{n+1}$ obtained by gluing the boundary of an $n+1$ disk e^{n+1} to S^n via a map $f\colon S^n \to S^n$ of degree k.
  (a) Use your knowledge of the homology of spheres to compute the homology of $M^n(k)$.
  (b) Prove that any map $g\colon M^n(k) \to M^n(k)$ has a fixed point.

(10) Suppose $f\colon A \to B$ and $g\colon B \to C$ are homomorphisms of abelian groups. Recall that the cokernel of f is $\operatorname{coker} f = B/f(A)$. Prove the existence of an exact sequence

$$0 \to \ker f \to \ker(g \circ f) \to \ker g \\ \to \operatorname{coker} f \to \operatorname{coker}(g \circ f) \to \operatorname{coker} g \to 0.$$