Swarthmore 2010 Honors Exam in Topology

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Do 6 of the 10 problems below as completely as you can, including at least one from each of the three parts.

Point-set topology

(1) Let $f\colon X \to Y$ be a bijective continuous map of topological spaces.
  (a) Find an example of such an f that is not a homeomorphism.
  (b) Find conditions on X and/or Y that force f to be a homeomorphism, and prove your answer.

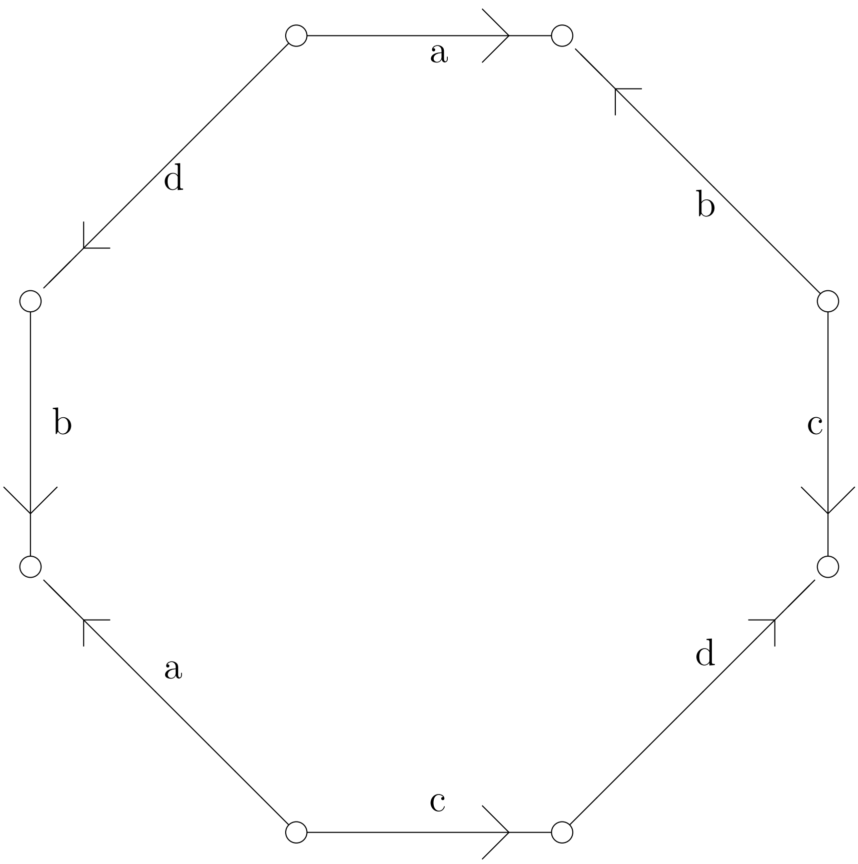
(2) Suppose $f\colon X \to Y$ is a map, not necessarily continuous, of topological spaces.
  (a) If f is continuous, prove that f preserves convergence of sequences. That is, prove that if (x_n) is a sequence in X converging to x, and f is continous, then the sequence $(f(x_n))$ converges to $f(x)$.
  (b) Find the best hypothesis you can on X that makes the converse to part (a) true, and prove it. That is, prove that, under your hypothesis on X, if f preserves convergence of sequences, then f is continuous.
  (c) Find an example of an f that preserves convergence of sequences, but which is not continuous.

(3) Prove that a connected metric space with more than one point and a countable dense subset must have exactly $\mathfrak{c}$ points, where $\mathfrak{c}$ is the cardinality of the real numbers.

Fundamental group and surfaces

(4) Suppose $f, g\colon S^1 \to S^1$ are continuous, f is homotopic to a constant map, and g is homotopic to the identity map. Show that there is some point $x \in S^1$ such that $f(x) = g(x)$. (You might start with the case where g is the identity map).

(5) Suppose we take an 8-sided polygon and identify the edges in pairs according to the labelling scheme below.
   (a) Prove that the resulting space K is a surface.
   (b) Calculate $\pi_1(K)$ and $H_1(K)$.
   (c) Identify this surface as one of the standard surfaces.



(6) Suppose we start with a path-connected space X and attach a 2-disk D^2 to X along a map $f\colon S^1 \to X$. That is, we form a new space Y as the quotient of the disjoint union of X and the unit disk D^2 in $\mathbb{R}^2$, obtained by identifying $x \in S^1 \subseteq D^2$ with $f(x) \in X$ for all $x \in S^1$.
   (a) Determine the fundamental group of Y in terms of the fundamental group of X and the map f.
   (b) Now suppose we attach a 3-disk D^3 to X instead, along a map $g\colon S^2 \to X$. Determine the fundamental group of the new space Z in terms of the fundamental group of X and g.

(7) Let X be a path connected, locally path connected, semilocally simply connected space. A path-connected covering space $p\colon E \to X$ of X is called **abelian** if it is regular and the group of covering transformations is an abelian group.
   (a) Prove that there is a universal abelian covering space $q\colon \widetilde{X} \to X$, in the sense that if $p\colon E \to X$ is any abelian covering space of X, then there is a covering map $r\colon \widetilde{X} \to E$ such that $pr = q$.
   (b) Describe this universal abelian covering space for the figure eight space $S^1 \vee S^1$.

Homology

(8) (a) Define the degree of a continuous map $f\colon S^n \to S^n$.
   (b) Calculate, with some proof, the degree of the map $f\colon S^n \to S^n$ defined by $f(v) = -v$.
   (c) Show that a map $f\colon S^n \to S^n$ of nonzero degree must be onto all of S^n.

(9) An n-dimensional **pseudomanifold** is a simplicial complex with the following three properties:
   1. Every simplex is a face of some n-dimensional simplex;

2. Every $(n-1)$-simplex is a face of exactly two n-simplices.
3. Given any two n simplices σ and τ, there is a finite sequence of n-simplices

$$\sigma = \sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_k = \tau$$

where σ_i and σ_{i+1} intersect in an $(n-1)$-simplex for each i.

If X is an n-dimensional pseudomanifold, there are two possibilities G_1 and G_2 for $H_n(X)$. Find them, with proof of course. What property distinguishes pseudomanifolds with $H_n(X) = G_1$ from those with $H_n(X) = G_2$?

(10) Prove the butterfly lemma. This says that if we have the commutative diagram of abelian groups and homomorphisms below,

$$\begin{array}{ccccc} A & \xrightarrow{i} & C & \xleftarrow{j} & D \\ {\scriptstyle f}\downarrow & & \| & & \downarrow{\scriptstyle g} \\ B & \xleftarrow[q]{} & C & \xrightarrow[p]{} & E \end{array}$$

(collapse the C's to see the "butterfly"), where both the diagonals

$$A \xrightarrow{i} C \xrightarrow{p} E$$

and

$$D \xrightarrow{j} C \xrightarrow{q} B$$

are exact at C, there is an isomorphism

$$\frac{\text{im } q}{\text{im } f} \cong \frac{\text{im } p}{\text{im } g}.$$