2009 Honors Examination: Topology

1. Define the terms "compact," "Hausdorff," and "closed" in the following theorem, and prove the theorem from these definitions. (You may want to set up a couple of lemmas along the way.)

Theorem. Any continuous map from a compact space to a Hausdorff space is closed.

2. Let X be the set of all sequences of real numbers, $x_0, x_1, \ldots$, with the topology determined by the metric given by

$$d(x,y) = \begin{cases} 1 & \text{if } |x_n - y_n| \geq 1 \text{ for some } n \\ \sup\{|x_n - y_n| : n \geq 0\} & \text{otherwise} \end{cases}$$

(a) First, check that this is indeed a metric.

(b) Let $x \in X$, and derive a condition on x which is equivalent to the statement that the function $\mathbb{R} \to X$ given by $t \mapsto tx$ is continuous. (Here $t(x_0, x_1, \ldots) = (tx_0, tx_1, \ldots)$.)

(c) Given $x \in X$, describe the connected component containing x.

3. Let S be a compact surface without boundary, let D^2 be the closed 2-disk, and let $D^2 \to S$ embed D^2 into S. If you like, you can take this embedding as a homeomorphism of D^2 with the standard 2-simplex, followed by the inclusion of the 2-simplex in a triangulation of S.

Remove the interior $\mathring{D}$ of the image of D^2. The result is a surface with boundary, and the boundary is equipped with a homeomorphism to S^1 regarded as the boundary of D^2.

Describe the Möbius band, and specify a homeomorphism of its boundary with S^1. Now attach a Möbius band to $S - \mathring{D}$ by identifying the boundaries according to how they have been identified with S^1. The result is the "blow-up" of S at the center point of $\mathring{D}$.

Discuss the resulting space. Is it a surface? What is its Euler characteristic (in terms of the Euler characteristic of S)? Is it orientable? What do you get if you start with $S = S^2$? How about if you start with $S = \mathbb{R}P^2$?

4. **(a)** What is the fundamental group of the circle? Sketch a proof, in a sentence or two.

(b) Let k be a positive integer. Please construct a simply connected space with a properly discontinuous action of the cyclic group C_k of order k.

(c) Please describe a space whose fundamental group is the free product of C_k with a free abelian group on two generators.

(d) Define the term "regular" as applied to a covering map, and give a specific example of a covering map which is not regular.

5. Suppose that

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d, "\cong"] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & A'_n \arrow[r] & B'_n \arrow[r] & C'_n \arrow[r] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

is a "ladder": a map of long exact sequences. So both rows are exact and each square commutes. Suppose also that every third vertical map is an isomorphism, as indicated. Prove that these data determine a long exact sequence

$$\cdots \longrightarrow A_n \longrightarrow A'_n \oplus B_n \longrightarrow B'_n \longrightarrow A_{n-1} \longrightarrow \cdots$$

6. Let A be a 2×2 matrix with integral coefficients. Let $\mathbb{T}^2$ be the torus, regarded as the quotient space of $\mathbb{R}^2$ by the translation action of the subgroup $\mathbb{Z}^2$. Explain why multiplication by A on $\mathbb{R}^2$ descends to a self-map of $\mathbb{T}^2$. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a continuous map which is homotopic to the map induced by A. Investigate when you can guarantee that f fixes a point in $\mathbb{T}^2$, in terms of invariants of the matrix A.

Is every continuous self-map of $\mathbb{T}^2$ of this form?