Swarthmore College
Department of Mathematics and Statistics

Honors Examination in Topology 2008

Instructions: Do at least one problem from each of the four parts of the exam. Try to complete at least 6 of the 12 problems. You may use without proof the basic theorems that you have learned, but be sure to state them carefully. You may also use a result from one problem in solving another, even if you didn't solve the first problem. If you have lots of time left over, feel free to solve other problems.

NOTATION

The following conventions for notation have been followed:
$\mathbb{Z}$ = the set (group, ring) of rational integers.
$\mathbb{Q}$ = the set (group, field) of rational numbers.
$\mathbb{R}$ = the set (group, field) of real numbers.
$\mathbb{C}$ = the set (group, field) of complex numbers.
S^{n-1} = the unit sphere in $\mathbb{R}^n$.
If X is a set and A a subset of X, then $X - A$ is the complement of A in X.

POINT SET TOPOLOGY

1. Suppose X is a set. The *finite-complement topology*, $\mathcal{T}_f$ is given by $\mathcal{T}_f = \{U \subset X \mid X - U \text{ is finite or } U = \emptyset\}$. (a) Show that $\mathcal{T}_f$ is a topology on X. (b) When is $(X, \mathcal{T}_f)$ a Hausdorff space? (c) If (X, d) is a metric space with the associated metric topology $\mathcal{T}_d$, show that $\mathcal{T}_f$ is coarser than $\mathcal{T}_d$, that is, $\mathcal{T}_f \subset \mathcal{T}_d$. (d) State the separation axiom T_1 and show that if $(X, \mathcal{T})$ is any topology on X, then $\mathcal{T}_f \subset \mathcal{T}$ if and only if $(X, \mathcal{T})$ is T_1.

2. The property of compactness is a generalization of the notion of a finite subset of a space. Take three properties of compact subsets of a topological space X and show how they generalize the same statement for a finite subset of X. (For example, the definition, or perhaps the Extreme Value theorem.) What is the analogous statement for the Lebesgue Covering Theorem when speaking of compact metric spaces?

3. A connected space X may contain points $x \in X$ called *cut points of order n*, defined by the property that $X - \{x\}$ has exactly n components. (a) Show that the property "x is a cut point of order n in X" is a topologically invariant property. (b) Construct a space containing a sequence of points $\{p_2, p_3, p_4, \ldots\}$ such that p_n is a cut point of order n. (c) Classify the alphabet up to homeomorphism (as shown) by analyzing cut points:

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

4. Quotients of the real line can be of all sorts. (a) Let $x \sim y$ be the relation on $\mathbb{R}$ where $x \sim y$ if $x - y \in \mathbb{Q}$. This is a group based relation with quotient given by $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Q}$. What is the quotient topology on $\mathbb{R}/\mathbb{Q}$ in this case? (b) Recall that if $A \subset X$, then the space X/A is the quotient of X by the relation $x \approx y$ if either $x, y \in A$, or $x \notin A$ and $x = y$. Show that the quotient $\mathbb{R}/(\mathbb{R} - [0, 1])$ is compact but not Hausdorff.

5. The so-called *Dunce Hat* is the quotient space of a 2-simplex with the edges glued according to the word $aa^{-1}a$. Use the Seifert-van Kampen theorem to compute the fundamental group of the Dunce Hat. What is the first homology H_1 of the Dunce Hat?

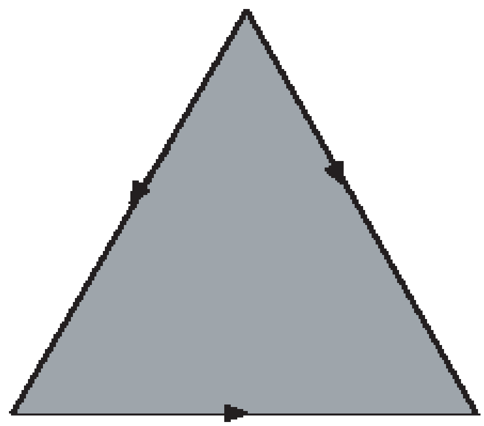

Figure 1: The Dunce Cap

6. Show that a continuous mapping $f\colon S^2 \to S^1 \times S^1$ is null-homotopic. (Hint: Try carefully using the lifting property of the universal covering space of $S^1 \times S^1$.)

7. Let $f_k\colon S^1 \to S^1$ denote the map of degree k given by $z \mapsto z^k$. Form the adjunction space $X_k = S^1 \cup_{f_k} e^2$ where we have attached a 2-cell to S^1 by attaching the boundary identifying z and $f_k(z)$. Use the Seifert-van Kampen theorem to compute $\pi_1(X_k)$. Classify all of the covering spaces of X_k.

8. A graph Γ on S^2 is a 1-dimensional simplicial complex with edges given by curves on S^2. Suppose that Γ is connected. Show that $S^2 - \Gamma$ is connected if and only if $V = E + 1$, where V is the number of vertices of Γ and E the number of edges.

9. *From the pages of Poincaré:* in 1885, he introduced the notion of the index of a flow on an orientable surface. Suppose that a surface S is compact, orientable, and without boundary, and there is a *flow* on S, that is, each point in S lies on a unique curve that satisfies some differential equation on the coordinates for the surface. The curves are all disjoint from one another, indexed by initial conditions. Assume that S is triangulated so that no edge in the triangulation is part of the flow. If Δ is a cycle on S (a closed path in the edges of the triangulation), then Poincaré defined the *index* of Δ to be

$$J(\Delta) = \frac{e(\Delta) - i(\Delta) - 2}{2}$$

where $e(\Delta)$ is the number of *exterior points* on the edges of Δ where the flow is tangent to the edge and outside Δ, $i(\Delta)$ is the number of *interior points* on the edges of Δ where the flow is tangent inside Δ. (a) Show that if Δ_1 and Δ_2 are cycles sharing an edge, then $J(\Delta_1) + J(\Delta_2) = J(\Delta_1 + \Delta_2)$, where the sum is the oriented sum of geometric cycles (the shared edge becomes interior). (b) Show that

$$\sum_{\Delta_i} J(\Delta_i) = -\chi(S),$$

where the sum is over all triangles of the triangulation of S. This is the 1885 version of the Poincaré index theorem.

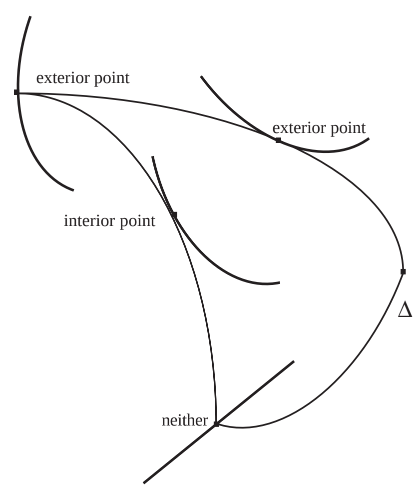

Figure 2: Points contributing to the index

10. Suppose K is a finite polyhedron and $F\colon K \times [0,1] \to K$ is a homotopy between the identity mapping on K and a continuous function $f\colon K \to K$. Show that $\chi(K) \neq 0$ implies that f has a fixed point. To which spheres does this result apply?

11. *From the pages of Borsuk:* in 1933 he proved the following statements :
i. For every continuous mapping $f\colon S^n \to \mathbb{R}^n$, there exists a point $\vec{x} \in S^n$ with $f(\vec{x}) = f(-\vec{x})$.
ii. For every antipodal mapping $f\colon S^n \to \mathbb{R}^n$ (that is, f is continuous and $f(-\vec{x}) = -f(\vec{x})$ for all $\vec{x} \in S^n$) there exists a point $\vec{x} \in S^n$ with $f(\vec{x}) = \vec{0}$.
iii. There is no antipodal mapping $f\colon S^n \to S^{n-1}$.
Show that these statements are equivalent.

12. Among the important constructions possible with simplicial complexes there is the *suspension* which is defined briefly for a simplicial complex K as

$$SK = K * S^0,$$

that is, SK is the *join* of K and the simplicial complex $S^0 = \{a, b\}$, the zero-sphere or boundary of the standard 1-simplex, Δ^1. More explicitly, SK has p-simplices given by $(x_0, \ldots, x_{p-1}, a)$ and $(x_0, \ldots, x_{p-1}, b)$, where $(x_0, \ldots, x_{p-1})$ is a $(p-1)$-simplex in K. Consider the algebraic mapping $s\colon C_{p-1}(K) \longrightarrow C_p(SK)$, given by

$$s\left(\sum_{i=1}^{n} \lambda_i (v_{0,i}, \ldots, v_{p-1,i})\right) = \sum_{i=1}^{n} \lambda_i [(v_{0,i}, \ldots, v_{p-1,i}, a) - (v_{0,i}, \ldots, v_{p-1,i}, b)].$$

Show that this mapping is a chain map, that is, $\partial \circ s = s \circ \partial$. Show that this induces a homomorphism $s_*\colon H_{p-1}(K) \to H_p(SK)$. We can compute the homology of SK by considering the subcomplexes $K * \{a\}$ and $K * \{b\}$ with union SK. Recall the Mayer-Vietoris sequence: If A and B are subcomplexes of X and $A \cup B = X$, then the inclusion mappings

$$i_1\colon A \cap B \hookrightarrow A,\ i_2\colon A \cap B \hookrightarrow B, \quad j_1\colon A \hookrightarrow X,\ j_2\colon B \hookrightarrow X,$$

determine a long exact sequence

$$\cdots \to H_p(A \cap B) \xrightarrow{i_{1*} \oplus i_{2*}} H_p(A) \oplus H_p(B) \xrightarrow{j_{1*} - j_{2*}} H_p(A \cup B) \to H_{p-1}(A \cap B) \to \cdots$$

Use this result to determine $H_*(SK)$ in terms of $H_*(K)$.