Swarthmore College
Department of Mathematics and Statistics

Honors Examination in Topology 2007

Instructions: Do 6 of the following 10 problems as thoroughly as you can. Include at least one problem from each of the four parts of the exam. You may use without proof the basic theorems that you have learned, but be sure to state them carefully. You may also use a result from one problem in solving another, even if you didn't solve the first problem. If you have lots of time left over, feel free to solve other problems.

Notation

The following conventions for notation have been followed:
$\mathbb{Z}$ = the set (group, ring) of rational integers.
$\mathbb{Q}$ = the set (group, field) of rational numbers.
$\mathbb{R}$ = the set (group, filed) of real numbers.
$\mathbb{C}$ = the set (group, filed) of complex numbers.
S^{n-1} = the unit sphere in $\mathbb{R}^n$.

Point Set Topology

1. Suppose $f, g\colon X \to Y$ are two continuous functions from X to Y. Define the *coincidence set* of f and g to be the subset of X

$$C(f, g) = \{x \in X \mid f(x) = g(x)\}.$$

   a) Show that if Y is Hausdorff, then $C(f, g)$ is a closed set in X.
   b) Suppose $A \subset X$ is a dense subset of X and $A \subset C(f, g)$. Deduce that $f = g$.
   c) Suppose $X = Y$ and $g = \mathrm{id}_X$, the identity mapping on X. Show that $C(f, \mathrm{id}_X)$ is the fixed set of f, $\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}$, and deduce that if X is Hausdorff, then $\mathrm{Fix}(f)$ is closed.

2. Let $S \subset \mathbb{R}^2$ denote the spiral given in polar coordinates by

$$S = \{(r, \theta) \mid 1 \le \theta < \infty \text{ and } r = (\theta - 1)/\theta\}.$$

   Let $A = \mathrm{cls}\, S$, the closure of S in $\mathbb{R}^2$. Show in detail that A is connected, but not path-connected.

3. If X is a topological space, then $\pi_0(X)$ denotes the set of equivalence classes of X under the relation $x \sim y$ if there is a continuous path $\lambda\colon [0, 1] \to X$ with $\lambda(0) = x$ and $\lambda(1) = y$.
   a) Suppose G is a topological group, that is, there is a binary operation on G, denoted $\mu\colon G \times G \to G$, which is continuous, makes G a group, and for which the mapping $g \mapsto g^{-1}$ is also continuous. Show that $\pi_0(G)$ is also a group.
   b) Let $\mathrm{Gl}_n(\mathbb{R})$ denote the multiplicative group of invertible $n \times n$ matrices with real entries. Show that the group $\pi_0(\mathrm{Gl}_n(\mathbb{R}))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. (Hint: Think about elementary matrices that carry out row operations and paths in $\mathrm{Gl}_n(\mathbb{R})$ between them. Or maybe think about the Gram-Schmidt Process.)

Fundamental Groups and Covering Spaces

4. The *suspension* ΣX of a topological space X is the quotient space of $X \times [0,1]$ under the equivalence relation given by the equality on points (x,t) for $0 < t < 1$ and for all $x, x' \in X$, we have $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$.
   a) Suppose that X is connected and path-connected. Show that the sets $U = \{[(x,t)] \mid t > 1/3\}$ and $V = \{[(x,t)] \mid t < 2/3\}$ are open in ΣX and that U and V are homotopy equivalent to a point.
   b) Show that $U \cap V$ is path-connected.
   c) Let $x_0 \in X$. Use a) and b) to show that $\pi_1(\Sigma X, [(x_0, 1/2)])$ is the trivial group.
5. If $p\colon \tilde{X} \to X$ is a covering space, then the lifting of loops in X to paths in $\tilde{X}$ leads to an action of the fundamental group $\pi_1(X, x_0)$ on the set $p^{-1}(\{x_0\})$.
   a) Describe the construction of this action in detail.
   b) The action of any group G on a set F leads to a homomorphism $\phi\colon G \to \mathrm{Sym}(F)$, where $\mathrm{Sym}(F)$ denotes the group of permutations of the set F. This homomorphism is defined by $g \mapsto (x \mapsto g \cdot x)$. Show that ϕ is injective.
   c) It follows that we have an injective homomorphism $\phi\colon \pi_1(X, x_0) \to \mathrm{Sym}(p^{-1}(\{x_0\}))$. Suppose $\tilde{X}$ is simply-connected, and the cardinality of $p^{-1}(\{x_0\})$ is 2. Deduce that $\pi_1(X, x_0) \cong \mathbb{Z}/2\mathbb{Z}$. What can you say if the cardinality of $p^{-1}(\{x_0\})$ is 3?

Surfaces

6. The classification theorem for surfaces shows that a (closed, compact, connected) surface may be represented as a quotient space of a polygon. This representation allows one to triangulate the surface, compute its fundamental group, etc. A surface is nonorientable if and only if it has a embedded Möbius band. Use the classification theorem to show that the connected sum of a projective plane with an orientable surface has an embedded Möbius band. And show that the canonical presentation of edge identifications for nonorientable surfaces contains an embedded Möbius band.
7. If you pluck a single point from S^2, then you get a space homeomorphic (via stereographic projection) to $\mathbb{R}^2$. Since $\mathbb{R}^2$ is convex, it is a contractible space and we can write $S^2 - \{x_0\} \simeq *$, the one-point space. In this problem, let's consider what happens when you remove a point from F, a compact, closed, connected surface. Show that, in general, $F - \{x_0\} \simeq S^1 \vee S^1 \vee \cdots \vee S^1$ for some number of circles (possibly none). Relate the number of circles in the bouquet to the genus of the surface when it is orientable, and generally to the Euler characteristic of F.

Algebraic Topology

8. Suppose that G is a topological group and $e \in G$ is the identity element. Then, in problem 3, we asserted that $\pi_0(G)$, the set of path components of G, is a group. We next show that $\pi_1(G, e)$ is a (left) $\pi_0(G)$-set, that is, there is an action of $\pi_0(G)$ on $\pi_1(G, e)$. Consider the mapping

$$\mu\colon \pi_0(G) \times \pi_1(G,e) \to \pi_1(G,e), \quad \mu([g],[\lambda]) = [g \cdot \lambda \cdot g^{-1}],$$

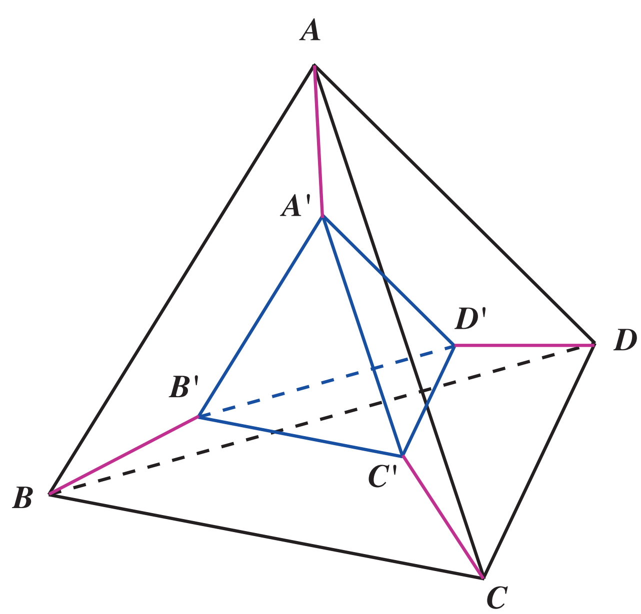
where $g \cdot \lambda \cdot g^{-1}(t) = g\lambda(t)g^{-1} \in G$.

a) Show that μ is well-defined, and that it satisfies the properties of a group action, that is, for all $[g], [h] \in \pi_0(G)$ and $[\lambda] \in \pi_1(G, e)$,

$$\begin{aligned} (1) \quad & [g] \cdot ([h] \cdot [\lambda]) = ([g][h]) \cdot [\lambda] \\ (2) \quad & [e] \cdot [\lambda] = [\lambda]. \end{aligned}$$

b) The topological group $O(2)$ consists 2×2-matrices with real entries and orthonormal columns. Show that $\pi_0(O(2))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and that the path component of the identity matrix Id is homeomorphic to S^1. It follows that $\pi_1(O(2), \mathrm{Id}) \cong \mathbb{Z}$. Thus the action in a) is algebraically an action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}$.

9. Consider the following 2-dimensional complex, denoted by K, given by taking a pair of tetrahedra, $ABCD$ and $A'B'C'D'$ with $A'B'C'D'$ smaller and inside $ABCD$ together with the edges AA', BB', CC', and DD', as well as the faces $ABB'A'$, $ACC'A'$, $ADD'A'$, $BCC'B'$, $BDD'B'$, and $CDD'C'$. The complex is pictured below.
   a) What is the Euler characteristic of this complex?
   b) Give a plausibility argument that this complex is simply-connected.
   c) Deduce from b) that $H_1(K) = \{0\}$ and from a) deduce the rank of $H_2(K)$.



10. Suppose K and L are finite simplicial complexes. Let $|K|$ and $|L|$ denote the realizations of K and L as topological spaces. Finally, let $[|K|, |L|]$ denote the set of homotopy classes of continuous functions from $|K|$ to $|L|$. Use the Simplicial Approximation Theorem to show that this set is countable. Show how this implies that $\pi_1(|K|)$ is a countable group if K is a finite complex.