Swarthmore College Mathematics
Honors Examination -Topology 2006

Divide your efforts somewhat equally among parts I, II and III. Unless otherwise specified, if using a known theorem, be sure to state it carefully before using it.

I. Point-set topology

1. For a topological space X, let CX denote the quotient of $I \times X$ by the relation $(0, x) \equiv (0, y)$ for all $x, y \in X$ where I is the unit interval.

a. Construct a bijection between CX where X is the set of natural numbers with the discrete topology and the set $Y \subset R^2$ (*with the subspace topology*) consisting of all points on the straight line segments from $(0, 1)$ to $(n, 0)$ for all natural numbers n. Determine if your bijection and/or its inverse are/is continuous.

b. Similarly compare Y with the set $Z \subset R^2$ (*both with the subspace topology*) consisting of all points on the straight line segments from $(0, 1)$ to $(1/n, 0)$ for all integers $n \geq 0$.

2.a. Consider the graph of $\sin(1/x)$ for $x > 0$ together with its limit points. Discuss the connectivity, local connectivity and path connectivity of this space.

2.b. Consider the graph of $\sin(1/x)$ for $\pi \geq x > 0$ together with an arc from $(\pi, 0)$ to $(0, 1)$ not intersecting the graph. Discuss the connectivity, local connectivity, path connectivity and fundamental group of this space.

3. Name conditions on a space X such that

a. every compact subset is closed.

b. every closed set is compact.

c. Give examples of spaces where a and/or b fail.

II. Algebraic topology of surfaces

4.a. The connected sum $X\#Y$ of two distinct compact surfaces without boundary X and Y is defined by removing an open disk from each of X and Y and then identifying the newly created boundaries. In terms of the van Kampen theorem, relate the fundamental groups of $X\#Y$ to those of X and Y.

4.b. What happens if the two distinct disks are removed from a single compact surface X? [Hint: find a way to decompose the resulting surface so you can apply van Kampen.]

5.a. Describe the boundary of an arbitrary compact surface with boundary. [Hint: The boundary may have more than one component.]

5.b. Relate the Euler characteristics of a compact surface without boundary to one with boundary. You may use the classification of compact surfaces without boundary, if it helps.

5.c. Describe the fundamental group of an arbitrary compact surface with boundary. Explain your reasoning.

6. Describe all possible covering spaces $X \to Y$ where Y is the Klein bottle. Which, if any, are orientable? Explain, quoting any theorems you need.

III. General algebraic topology

7. Compute the relative homology of the pair (X, A) where X is a pretzel (Philadelphia style = solid 2-hole torus) and A is the boundary of X, using (without proof) the basic properties of homology and your knowledge of the homology of a circle and of a point.

8.a. State precisely the conditions for and the exact sequence in homology of a pair (X, A) and the exact Mayer-Vietoris sequence.

8.b. Assuming both are valid and using other properties of homology, show that either implies the other.

9. Given a chain complex $C = \{C_i, d_i\}$ of free modules over the integers Z, where $d_i : C_i \to C_{i-1}$, the homology group H_i is defined as the quotient $Ker\ d_i/Im\ d_{i+1}$.

Cohomology is then defined by taking the duals $C^i = Hom(C_i, Z)$ and defining $\delta : C^i \to C^{i+1}$ by $\delta h = h \circ d$ where $d : C^{i+1} \to C^i$ and $h : C_i \to Z$.

a. Show $\delta \circ \delta = 0$.

b. Compute $H^i(P^2)$ and compare to $H_i(P^2)$.

c. Compute $H^i(P^3)$ and compare to $H_i(P^3)$.