Honors Exam in Topology
Swarthmore College
Department of Mathematics and Statistics, 2002

The exams contains 12 problems. Do at least 2 problems from each section. In each section, the harder problems are first, and will be marked appropriately.

GENERAL TOPOLOGY:

I.1 Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous bijection.
  (a) Suppose X and Y are both compact and Hausdorff. Prove that f is a homeomorphism.
  (b) Can you remove a hypothesis on both X and Y and still have a theorem? Explain any necessary adaptations of the proof.

I.2 Let $X \subseteq \mathbf{R}^2$ be the *Hawaiian earring*, i.e. the union of the circles of radius $1/n$ around the point $(1/n, 0)$ for $n = 1, 2, \ldots$. Let Y be the union of circles of radius n around $(n, 0)$.
  (a) Construct a continuous bijection from one of these spaces to another.
  (b) Is this a homeomorphism?

I.3 Prove that any compact Hausdorff space is normal.

SURFACES:

II.1 (a) Show that any triangulation of a closed surface contains at least 4 two-simplices.
  (b) Give an example of a triangulated closed surface with exactly 4 two-simplices.

II.2 State the classification theorem for compact surfaces. Denote the Klein bottle by K^2 and the projective plane by P^2. Identify, using any techniques you need from the classification of surfaces, $K^2 \# K^2$, $P^2 \# P^2$ and $P^2 \# K^2$

II.3 (a) Define the euler characteristic of a simplicial complex.
  (b) Derive a formula for the connected sum of two surfaces in terms of the euler characteristics of the individual surfaces.
  (c) Repeat the previous exercise for two n-manifolds.

FUNDAMENTAL GROUP:

III.1 Prove the Nielsen-Schreier theorem (any subgroup of a free group is free).

III.2 The Fundamental Theorem of Algebra states that any non-constant polynomial over the complex numbers, $\mathbf{C}$, has a zero. Prove this theorem.

III.3 A *graph* is a simplicial complex with only 0-cells and 1-cells (thought of as a collection of *vertices* and *edges*). If Γ is a graph, and $\chi(\Gamma) < 1$, what can one say about Γ?

HOMOLOGY:

IV.1 Denote real projective m-space by P^m, so that P^m is the quotient space obtained by identifying antipodal points of the m-sphere, S^m.

(a) Show any self map of P^{2n} has a fixed point.

(b) Exhibit a self map of P^{2n+1} with no fixed point.

IV.2 (a) Define a *simplicial map*.

(b) Prove that if $f : |K| \to |L|$ has a simplicial approximation s, then f is homotopic to s.

(c) State the simplicial approximation theorem.

(d) Prove that if $k < n$, K a simplicial complex of dimension k, then any map $f : |K| \to S^n$ is homotopic to a constant map.

IV.3 If C_* is a chain complex of vector spaces over K (each group is a vector space over K, and the differentials (the boundary maps in the chain complex) are K-linear), prove that $H_i(C_*)$ is a K-vector space for each i.