Swarthmore College
Department of Mathematics and Statistics
Topology Honors Examination 2000

Do as many questions as you can; if you get stuck go on to something else. You may use a result from one problem in solving another,even if you did not solve the first problem. You may also use without proof:

· Any facts you know about the homology groups of balls or spheres.

· The homotopy invariance of the fundamental group, homology groups, or of the Euler characteristic.

· The least upper bound property of the real numbers.

· The Heine-Borel theorem. (Characterizes compact subsets of Euclidean space.)

· The Lefschetz fixed point theorem. (Be sure to state it carefully.)

1) Prove that any open, connected subset of Euclidean space $\Re^n$ is path connected.

2) If A is a linear map $\Re^n \to \Re^n$, the norm of A is defined by

$$\text{Norm}(A) = \underset{x \in \Re^n, x \neq 0}{Sup} \frac{||Ax||}{||x||}$$

Show that for any linear map A, the norm is finite. Do not mention matrices. The only things you should need from algebra are:

$||ax|| = a||x||$ if $a > 0$

A is continuous.

$Aax = aAx$ if $a \in \Re$

The norm $||x||$ on $\Re^n$ is continuous.

$||x|| \neq 0$ if $x \neq 0$.

3) Let X be Hausdorff. Show that if K is a compact subset of X, and x a point of X not in K, then there are disjoint open sets U,V in X with K contained in U and x contained in V.

4) a. State the classification theorem for compact surfaces. (All surfaces are assumed connected.)

b. Explain how you can be sure that no two of the surfaces on your list are homeomorphic.

c. Let T be the connected sum of 5 tori and P the connected sum of 3 projective planes. Three small disks are cut from each of the 2 surfaces, and each boundary circle from T is glued to a boundary circle from P. Where does the resulting surface fit into your classification?

5) Let $S^1 = \Re/Z$. A continuous map $f : S^1 \to S^1$ lifts to a continuous map $F : \Re \to \Re$. The lift is not unique but if x is in $\Re$ the degree

$$d = F(x+1) - F(x)$$

does not depend on the lift. Show that

a) The degree is an integer.

b) The degree does not depend upon the choice of x.

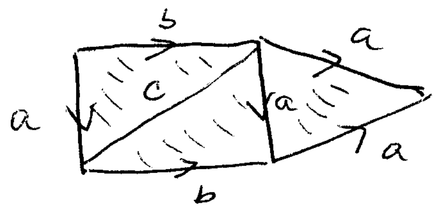
c) Two continuous maps $f, g : S^1 \to S^1$ with the same degree are homotopic. (You should be able to write down a map which descends to a homotopy; explain briefly why the construction would not work if the degrees were not equal.)

6) Consider the following statement:

Let X be a topological space which is the union of two path-connected, simply connected, open subsets U and V. Assume $U \bigcap V$ is nonempty. Then X is simply connected.

If true, outline the proof of the part of the Seifert-Van Kampen theorem from which it follows. If false, give a counterexample, say what part of the hypothesis of Seifert-Van Kampen is missing, and explain how the proof of the relevant part of the theorem fails when this hypothesis is missing.

7) Let X be the quotient space obtained from identifying the edges of three solid triangles indicated below. Compute the homology groups of X.



Is there a compact surface with these homology groups? Explain briefly.

8) For each pair (X, A) write out the long exact homology sequence. Give the isomorphism type (e.g. $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$) for each group in the sequence. Give a rough description of generators and (if applicable) relations for each nonzero group $H_k(X, A)$. (There should be 3 groups in all you need to describe this way.)

a. X= Cylinder $S^1 \times [0,1]$, A = Boundary $(X) = S^1 \times \{0,1\}$

b. X= Mobius band, A= Boundary $(X) = S^1$.

9) Find a fixed-point free map from the circle S^1 to itself, and show that every fixed-point free self map of the circle is homotopic to this one.