Swarthmore College
Department of Mathematics and Statistics
Honors Examination 1999

TOPOLOGY

NOTE: You may use without proof
1) any facts you know about the homology groups of spheres or balls
2) the homotopy invariance of the fundamental group, homology groups, or the Euler characteristic
3) the Least Upper Bound property of the real numbers
4) the Seifert-Van Kampen theorem (but state it clearly before you use it).

1) Here is a theorem used in beginning calculus:

> If f: [0,1] --> R is continuous, then f is bounded and achieves its maximum value at some point x in [0,1].

This theorem is a consequence of two theorems in topology. (One has to do with the topology of the real line, and the other applies to general topological spaces.)

a) State the two theorems.

b) Prove one of the two theorems.

c) One also uses in calculus the fact that f attains all values between its maximum and minimum values. What is the general topological fact of which this is a manifestation?

2) a) What is the projective plane? What is its fundamental group? You can use a general theorem if you state it and include details of how it applies in this case.

b) Let X be a topological space which is the union of 2 open sets U and V. Assume that U, V, and the intersection of U and V are all path connected, and that the intersection is also simply connected. Take a base point in the intersection. Show that the inclusion of the fundamental group of U in the fundamental group of X is an injection (1-1).

c) Sketch the universal cover of the figure eight.

3) a) State the classification theorem for compact surfaces. (All surfaces are assumed connected and without boundary.)

b) Are two compact surfaces with the same Euler characteristic necessarily homeomorphic? (If you know a formula for the Euler characteristic of a surface, you may use it here without proof.)

c) Are two compact surfaces with the same homology groups necessarily homeomorphic? (Note it may not be necessary to write down the homology groups of all surfaces in order to answer this. It may help to use b).)

d) Are two compact surfaces with the same fundamental group necessarily homeomorphic? (Note it may not be necessary to write down all the groups. It may help to use c).)

e) Let T be the connected sum of 3 tori, and P the connected sum of 5 projective planes. Three small disks are cut from each of the two surfaces, and each boundary circle from T is glued to a boundary circle from P. What is the color of the busdriver's eyes? No, seriously, where does the resulting surface fit in your classification? Explain briefly.

4) Here are 2 maps from the unit square to itself:

f: (x,y) --> (y,x)   (x and y are in [0,1])

g: (x,y) --> (1-y,x).

The maps f and g induce maps F and G on the torus. Compute the Lefschetz number of F, and that of G. (Brief explanations will suffice.) Is F homotopic to a fixed point free map? Is G? You should be able to answer definitively in each case. You may use the Lefschetz fixed point theorem here.

5) State and prove the Brouwer fixed-point theorem for a ball in n-dimensional Euclidean space. (You may NOT use the Lefschetz theorem on this one.)

6) a) Compute the homology groups of the Klein bottle from a triangulation.

b) (Short answer; no explanations whatever are required) Figure A is a commutative diagram of abelian groups in which the arrows represent group homomorphisms. Assume the rows are exact. What conditions on the maps f and h will ensure that g is injective? Try to give necessary and sufficient conditions.

(diagram commutes!)

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \to & D & \to & E & \to & F & \to & 0 \end{array}$$

FIGURE A

c)(Extra credit) (Short answer) Figure B is another commutative diagram of abelian groups and homomorphisms with the rows exact. What conditions on the maps a,b,d, and e will ensure that c is injective?

(diagram commutes!)

$$\begin{array}{ccccccccc} A & \to & B & \to & C & \to & D & \to & E \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d & & \downarrow e \\ F & \to & G & \to & H & \to & I & \to & J \end{array}$$

FIGURE B

d)(Short answer) In Figure C, X is the union of 3 surfaces (with boundary) and A is the boundary of X. Thus A is the union of 4 circles. The homology groups of X and of A, and the relative homology groups of the pair (X,A) are all free abelian groups. Write out the exact homology sequence, and identify all the groups. (You need to give the rank of each group.)

FIGURE C