Swarthmore College Honors Exam in Topology, 1998

1. Let A be a subset of a topological space X.
   a. Define the closure of A, and give two examples.
   b. Prove that if there exists a sequence of points of A which converges to a point $x \in X$, then x is in the closure of A.
   c. Prove that the converse of (b) holds if X is metrizable.
   d. Give an example in which the converse of (b) does not hold. Explain.

2. Let X be an uncountable set with the finite complement topology. Thus the nonempty open subsets of X are the sets with finite complements. Give a reason for your answer to each of the following.
   a. Is X connected?
   b. Is X compact?
   c. Is X Hausdorff?
   d. Is X metrizable?

3. Suppose that E and B are path-connected spaces, and $p : E \to B$ is a map with unique path-lifting. This means that for any points x_0 and b_0 satisfying $p(x_0) = b_0$, and for any path ω in B which begins at b_0, there exists a unique path $\tilde{\omega}$ in E which begins at x_0 and satisfies $p \circ \tilde{\omega} = \omega$. Let b_0 and b_1 be any two points of B. Prove that $p^{-1}(b_0)$ and $p^{-1}(b_1)$ have the same cardinality; i.e., they can be placed in 1-1 correspondence. (Remark: p is not necessarily a covering projection.)

4. Let T denote the torus, and x and y distinct points of T. Determine $\pi_1(T)$, $\pi_1(T-\{x\})$, and $\pi_1(T - \{x, y\})$. Give reasons for your answers.

5. Determine the homology groups of spheres, and use these to prove the Brouwer Fixed Point Theorem. Give as much detail as you can. However, you may use without proof any of the major properties of homology groups such as exact sequences and homotopy invariance. You may also use all basic results of simplicial topology, if you wish. Your essay should include a statement of the two results which you are proving.

6. State the classification theorem for compact surfaces. Define the examples that you use. Explain how you can assert that your examples are not homeomorphic to one another. You may use any general results that you wish here, but you should show how they apply to your particular examples.