SWARTHMORE COLLEGE
Department of Mathematics and Statistics
Honors Examination

8 May 1997

Topology

DIRECTIONS: Do two problems from part I, two problems from part II, and as much of part III as you can. If you get stuck on III, or if you have extra time, do some more of I and II.

I. POINT-SET TOPOLOGY.

I.1. For (a) and (b), assume that X is a Hausdorff space.

(a) Show that every finite subset of X is closed.

(b) Let A be a subset of X. Use part (a) to show that x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

(c) You can actually prove (a) and (b) with a related, but weaker, condition on X than "Hausdorff"; what is that condition?

I.2.

(a) Fix a set I, and let $\{X_\alpha : \alpha \in I\}$ be a collection of topological spaces. What is the coarsest topology on the set $\prod X_\alpha$ relative to which each projection map π_β is continuous? Prove that your answer is correct.

(b) Show that if X and Y are connected, then so is $X \times Y$ (in the product topology).

I.3.

(a) Let X be a metric space. Show that if $A \subseteq X$ is compact, then A is closed and bounded. Give an example to show that the converse is false.

(b) Show that if X and Y are compact, then so is $X \times Y$ (in the product topology).

II. SURFACES, ALGEBRAIC TOPOLOGY.

II.1. State and prove (using the fundamental group) either the Brouwer fixed point theorem or the fundamental theorem of algebra.

II.2. Let S^m denote the unit m-sphere.

(a) Prove that if $m > 1$, then every map $S^m \longrightarrow S^1$ is homotopic to a constant map.

(b) Prove that if $m < n$, then every map $S^m \longrightarrow S^n$ is homotopic to a constant map.

II.3. Here is a commutative diagram of abelian groups:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Suppose that the rows are short exact sequences.

(a) Show, by example, that the map f need not be zero.
(b) What *can* you say about the map f in general?

II.4. Consider the "one-point union" of S^1 and S^2 (say, a basketball with a rubber band attached to it).

(a) What is its universal cover?
(b) Compute π_1 of this space.
(c) Compute H_n of this space, for all $n \geq 0$.

II.5. **Theorem.** *The Euler characteristic of a compact, path-connected, triangulable topological group is zero.*

(a) List all of the compact surfaces (without boundary), and give their Euler characteristics. According to the theorem, which of these can be topological groups? Which surfaces actually are topological groups?
(b) For which n can the n-sphere possibly be a topological group? For which n can you (easily) prove that S^n is a topological group?
(c) Prove the theorem. (Apply the Lefschetz fixed point theorem to the translation map $h \mapsto gh$, for some fixed $g \in G$.)

III. The Projective Plane.

III.1. Let P^2 denote the projective plane. Do as many of the following as you can; you may use whichever definition of P^2 is most convenient for each part.

(a) Is P^2 compact? Path-connected? Hausdorff? Second countable? Prove your answers.
(b) What is the universal covering space of P^2? Use this to show that $\pi_1(P^2, y) \cong \mathbf{Z}/2$, for some $y \in P^2$.
(c) What is the homology of P^2? (You don't have to give a detailed proof—just explain the main points.)
(d) What is the Euler characteristic of a connected sum of n projective planes? Give a brief explanation.
(e) Show that every map $P^2 \longrightarrow T^2$ is homotopic to a constant map (where T^2 denotes the torus: $T^2 = S^1 \times S^1$).
(f) Show that for any continuous map $f : P^2 \longrightarrow P^2$, there is a point $x \in P^2$ so that $f(x) = x$.
(g) Suppose that X is a path-connected topological space with $\pi_1(X, x) \cong \mathbf{Z}/2$ for some basepoint $x \in X$. Show that there is a map $P^2 \longrightarrow X$ (sending y to x) which induces an isomorphism on π_1.