SWARTHMORE COLLEGE
Department of Mathematics and Statistics
Honors Examination

13 May 1996
8:30–11:30

Topology

DIRECTIONS: Please do at least one problem from each of the four sections; try to do at least six problems altogether. If you are asked to provide a counterexample for something, explain briefly why it's a counterexample—you don't have to give all of the details.

I. POINT-SET TOPOLOGY.

I.1.

(a) Prove that X is Hausdorff if and only if the set $\Delta = \{(x,x) \in X \times X\}$ is closed in $X \times X$.

(b) If X is connected, must X be path connected? If X is path connected, must X be connected? For each question: if you answer yes, prove it; if no, then provide a counterexample.

I.2.

(a) Suppose that $f\colon X \to Y$ is a continuous bijective map with X compact and Y Hausdorff. Must f be a homeomorphism? If so, prove it. If not, provide a counterexample.

(b) Do the same thing, but instead assuming that X is Hausdorff and Y is compact.

I.3.

(a) A collection $\mathcal{C}$ of subsets of X is said to satisfy the *finite intersection condition* if for every finite subcollection $\{C_1, \ldots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap \ldots \cap C_n$ is nonempty. There is a theorem that says that a space X is compact if and only if (something to do with the finite intersection condition). Clarify the part in parentheses, and prove the theorem.

(b) Use part (a) to prove that if X is a nonempty compact Hausdorff space such that every point of X is a limit point of X, then X is uncountable.

II. CLASSIFICATION OF SURFACES.

II.1. Suppose that M is a compact triangulated surface. Let v, e, and t denote the numbers of vertices, edges, and triangles in the triangulation; let χ denote the Euler characteristic of M. Show that

$$3t = 2e, \qquad e = 3(v - \chi),$$
$$\frac{v(v-1)}{2} \geq e, \qquad v \geq \tfrac{1}{2}(7 + \sqrt{49 - 24\chi}).$$

What are lower bounds for v, e, and t when M is the sphere? The torus? The projective plane? Can you give triangulations realizing these lower bounds?

II.2. Use the Euler characteristic to prove that there are only five regular polyhedra.

II.3. Describe the classification of compact surfaces, and sketch the proof.

III. The fundamental group.

III.1.
(a) Let X and Y be topological spaces. How is the fundamental group of $X \times Y$ related to the fundamental groups of X and Y?
(b) Compute the fundamental group of the *solid torus* $S^1 \times B^2$, where

$$B^2 = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 \leq 1\}$$

is the unit disk. (Use part (a) or not, as you wish.)

III.2.
(a) Let X be the disjoint union of a circle and a disk. What is the fundamental group of X?
(b) State Van Kampen's theorem in as general a form as you can; use it to compute the fundamental group of S^n for $n > 1$.

III.3.
(a) Fix an integer $n \geq 0$, and let P^n be the quotient space of S^n obtained by identifying each $x \in S^n$ with its antipode. Show that the map $S^n \to P^n$ is a covering map.
(b) Use this to compute $\pi_1(P^n, y)$ for $n \geq 0$ (for some point $y \in P^n$).

III.4. Suppose that G is a topological group with identity element e. Must $\pi_1(G, e)$ be abelian? Prove or give a counterexample.

IV. Homology.

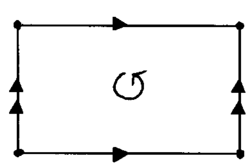
IV.1.
(a) If K is a simplicial complex with subcomplexes L and M so that $L \cup M = K$, then the *Mayer-Vietoris sequence* is the following long exact sequence:

$$\cdots \longrightarrow H_n(L \cap M) \xrightarrow{i_* \oplus (-j_*)} H_n(L) \oplus H_n(M) \xrightarrow{k_* + \ell_*} H_n(K) \longrightarrow H_{n-1}(L \cap M) \longrightarrow \cdots.$$

(Here, i_*, j_*, k_*, ℓ_* are all maps induced by the obvious inclusions.) Prove that this is exact.
(b) Use it to compute the homology groups of S^n for $n \geq 1$.

IV.2. State the Lefschetz fixed-point theorem and use it to prove that the projective plane has the fixed-point property. Try to give some details of the relevant homology calculations, rather than just stating them.

IV.3. Consider the following representation of the torus:



Pretend that this is a simplicial complex with 1 vertex, 2 edges, and one "triangle" (well, it looks more like a rectangle), with orientations as pictured. Now compute its homology groups. Do you get the right answer? Would you get the right answer if this had been some other surface? Explain what's going on here. (You don't have to give all of the details, just hit the high points.)