SENIOR HONORS EXAM IN TOPOLOGY

Swarthmore College

15 May 1993

INSTRUCTIONS:

Please do at least 6 of the following problems as thoroughly as you can. Choose at least two problems from each of the following three sections:

SECTION I:

(1). Show that if A is a closed subspace of a compact Hausdorff space X, then

(a). A is compact, and

(b). X/A is Hausdorff.

(2). Show that the unit interval [0,1] is connected in the usual Euclidean topology.

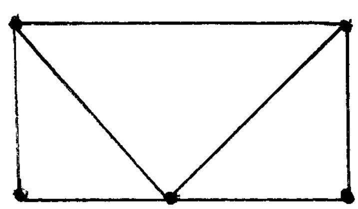
(3). (a). Let Q denote the rational numbers and let a,b,c and d be irrational numbers with $a < b$, and $c < d$. Consider the intervals $A = (a,b)$ and $B = (c,d)$ and show that $A \cap Q$ is homeomorphic to $B \cap Q$.

(b). Using part (a) show that $[0,1) \cap Q$ is homeomorphic to $(0,1) \cap Q$.

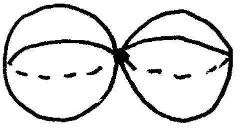
(4). Prove that the product of two compact spaces is compact (in the product topology).

SECTION II:

(5). Compute the homology of the one-dimensional simplicial complex whose 0-simplices and 1-simplices are pictured below.



(6). (a).Compute the homology of the space Y pictured below where Y is the one point union of the 2-sphere with itself.



(b). Let f be a continuous map from Y to itself. Show that f has a fixed point. That is, show that there is a point p in X such that $f(p) = p$.

(7). Let P(2) denote the space of all monic polynomials of degree two over the complex numbers in the variable z. Thus P(2) is the space of polynomials

$$\{ z^2 + az + b \mid a \text{ and } b \text{ are complex numbers} \}$$

which is topologized as a subspace of Euclidean space given by C x C.

(a). Show that P(2) is contractible.

(b). Let D(2) denote the subspace of P(2) given by the polynomials with exactly 2 distinct roots. Show that D(2) is homeomorphic to the cartesian product of a circle and 3-dimensional Euclidean space.

(c). Show that D(2) cannot be a retract of P(2).

(8). Let X denote the subspace of a product of two circles given by

$$\{(x,y) \mid x \text{ and } y \text{ are unit complex numbers, } x \neq y, \text{ and } x \neq -y\}.$$

Define $f : X \longrightarrow X$ by $f(x,y) = (y,x)$. Is f homotopic to the identity $1: X \longrightarrow X$? Please justify your answer.

<u>SECTION III:</u>

(9). Let S(3) denote a closed, orientable surface of genus 3.

(a). Compute the fundamental group of S(3).

(b). Compute the first homology group of S(3).

(c). Show that the abelianization of the fundamental group of S(3) is isomorphic to the first homology group of S(3).

(10). Let G be a finite group. Asssume that G acts continuously on a path connected Hausdorff space X and that G acts without fixed points (That is, if g(x) = x for some x then g = 1.). Thus G is a group of deck transformations for X.

(a). Show that there is an exact sequence

$$1 \longrightarrow \pi_1(X) \longrightarrow \pi_1(X/G) \longrightarrow G \longrightarrow 1.$$

(b). Give an example of this exact sequence when G = Z/3Z where the fundamental group of X/G has no elements of finite order.

(11). Consider the group G = Z/2Z and define an action of this group as deck transformations on a product of two circles V = S x S by the formula $g(x,y) = (-x, \overline{y})$ for g the non-identity element in G and unit complex numbers x and y. (Here $\overline{y}$ denotes the complex conjugate of y.) Let W be the quotient space V/G. Use the results in problem (10) to answer the following questions:

(a). Is W orientable ?

(b). What is the genus of W ?

(c). Identify the surface W in terms of the classification of closed surfaces.

(d). What is the universal cover of this surface ?

(12). Let K denote the Klein bottle.

(a) Determine the fundamental group of the surface K.

(b) Find the abelianization of this group

(c) Find the first homology group of the surface K.

(d) Show that the abelianization of the fundamental group is isomorphic to the first homology group of the surface.