Swarthmore College
Department of Mathematics and Statistics
Honors Examinations in Geometry 2019

Instructions: Do as many of the following problems as thoroughly as you can in the time you have. Include at least one problem from each of the three parts of the exam.

Part I: Curves

1) The *tangent indicatrix* of a regular closed space curve $\gamma : [0,1] \to \mathbb{R}^3$ is the curve $T : [0,1] \to S^2$ such that $T(t) = \frac{\gamma'(t)}{|\gamma'(t)|}$ for every $t \in [0,1]$.

   a) Show that the arc length of T is equal to the total curvature of γ.

   b) Show that T intersects every great circle on S^2.

   c) Conclude that the total curvature of γ is at least 2π.

2) Let $\gamma : [0,1] \to \mathbb{R}^3$ be a regular curve. Given a point P on the image of γ, show that there is a unique helix passing through P whose curvature and torsion coincide with those of γ at P.

3) Let X be the curve in $\mathbb{R}^2$ defined by the equation $x^4 - x^2 + y^2 = 2$. Show that the derivative of the signed curvature of X vanishes at at least four points. Where are these points located?

4) Consider the subset $X \subseteq \mathbb{R}^3$ defined by the equations

$$\begin{cases} x^2 + y^2 = 1 \\ x^2 + z^2 = 1 \\ yz \geq 0 \end{cases}$$

   a) Show that X is a closed curve and find an explicit parametrization.

   b) Calculate the curvature of X.

   c) Calculate the torsion of X.

Part II: Surfaces

5) Consider the hypersurface $X \subseteq \mathbb{R}^3$ defined by the equation $z = x^2y^2$.

   a) Calculate the first fundamental form and the second fundamental form of X.

   b) Find the hyperbolic points of X.

   c) What is the image of the Gauss map?

6) A *Liouville Surface* is a surface with metric of the form

$$ds^2 = (U(u) + V(v))(du^2 + dv^2)$$

   for some functions $U, V : \mathbb{R} \to \mathbb{R}$.

   a) Show that every surface of revolution is a Liouville surface.

   b) Express the Christoffel symbols of a Liouville surface in terms of U and V.

   c) Express the Gaussian curvature of a Liouville surface in terms of U and V.

7) The *Henneberg surface* X can be represented in $\mathbb{R}^3$ using the parametrization

$$x(u,v) = 2\sinh(u)\cos(v) - \frac{2}{3}\sinh(3u)\cos(3v)$$
$$y(u,v) = 2\sinh(u)\sin(v) + \frac{2}{3}\sinh(3u)\sin(3v)$$
$$z(u,v) = 2\cosh(2u)\cos(2v).$$

a) Calculate the Gaussian curvature of X.

b) Show that X is a minimal surface.

c) Show that X is non-orientable.

8) Let X be a compact connected orientable surface without boundary and positive Gaussian curvature. Show that any two simple closed geodesics on X must intersect.

Part III: Manifolds

9) Let S^n_r be the standard sphere of radius r. Calculate the scalar curvature of $X_{r,r'} = S^2_r \times S'^3_r$. Are there values of r, r' for which $X_{r,r'}$ is Einstein (i.e. the Ricci tensor is proportional to the metric)?

10) A Riemannian manifold X is homogeneous if given $x, y \in X$ there exists an isometry $f : X \to X$ such that $f(x) = y$. Show that homogeneous Riemannian manifolds are complete.

11) Let J be a $(1,1)$-tensor such that $J^2 = -\mathrm{Id}$. Show that the expression

$$N(X,Y) = [J(X), J(Y)] - J([J(X), Y]) - J([X, J(Y)]) - [X, Y]$$

defined for all vector fields X, Y defines a tensor (known as the *Nijenhuis tensor)*.

12) Calculate the isometry group of the hyperbolic half-plane. What can you say about the isometry group of the three-dimensional hyperbolic half-space?