Honors Examination in Statistics

Department of Mathematics and Statistics
Swarthmore College

Instructions: This exam consists of a total of four questions. Number the questions clearly in your work and start each question on a new page. You must show your work to make it clear how you obtained your answers. Answers without any work may lose credit even if they are correct, and will receive no credit if incorrect.

This is a closed-book three-hour exam. You may not refer to your notes or textbooks.

1. In a town of $N+1$ inhabitants, a person tells a rumor to a second person, who in turn repeats it to a third person, and so on. At each step the recipient of the rumor is chosen at random from the N inhabitants available (so, for example, the second person may repeat the rumor to the first person).

   (a) Find the probability that the rumor will be told k times without returning to the originator.

   (b) Find the probability that the rumor will be told k times without being repeated to any person.

   (c) Let X be the number of times the rumor is told until it returns to the originator. Determine the probability mass function of X.

   (d) Let Y be the number of distinct inhabitants that know the rumor after k calls. Find $\mathrm{E}(Y)$.
   (Hint: Define Y_i to be 1 if inhabitant i knows the rumor after k calls, and 0 if not. Find $\mathrm{E}(Y_i)$ for each i, and then relate Y to the Y_i to find $\mathrm{E}(Y)$.)

2. Suppose X is uniformly distributed on $[0, 1]$, and, conditional on X, Y is uniformly distributed on $[0, X]$.

   (a) Find $\Pr(Y > 0.5 | X = 0.8)$.

   (b) Determine the joint density function of X and Y. Be specific about the domain of values where the function is nonzero.

   (c) Find $\mathrm{Cov}(X, Y)$.

   (d) Let $W = X - Y$. Determine the probability density function of W.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $\mathrm{N}(\mu, \sigma^2)$.

   (a) Derive the maximum likelihood estimator (MLE) $\hat{\sigma}^2$ for σ^2.

   (b) Show that the MLE $\hat{\sigma}^2$ is a consistent estimator for σ^2.

   (c) Find an unbiased estimator, $\tilde{\sigma}^2$, for σ^2, as a function of the MLE, $\hat{\sigma}^2$.

   (d) Obtain the Cramer-Rao lower bound for the variance of an unbiased estimator for σ^2.

4. It is claimed by the seller of fishing rights that a lake contains at least N fish (N large). To investigate this claim, part of the lake is netted, and m captured fish are tagged and returned to the lake. Subsequently, when the tagged fish have distributed themselves over the lake, n fish are captured.

   (a) If R is the number of tagged fish out of n selected, what is the exact probability distribution of R?

   (b) As N becomes large relative to n, what is the approximate distribution of R?

   (c) Suppose $R = r$ tagged fish are observed out of the n captured. The seller's claim is rejected if r/n is greater than some number k. Show how to determine k such that the probability of falsely rejecting the claim is not more than 0.1 (assume that m and n are small relative to N).