# SWARTHMORE COLLEGE REAL ANALYSIS HONORS EXAM 2026

**Instructions:** Do as many problems, or parts of problems, or special cases of problems, as you can. Justify all answers. You may quote any standard result as long as it is not essentially what you are being asked to prove.

## Real analysis I

**1.** For each of the following, give an example or prove that no such example exists.
 (a) A function that is not the derivative of any other function.
 (b) A differentiable function with a discontinuous derivative.
 (c) A non-constant function with derivative equal to 0 everywhere.
 (d) A series for which the Root Test applies but the Ratio Test does not.

**2.** (a) Prove the following theorem: Let $A$ be a subset of $\mathbb{R}$. If $\{f_n\}$ is a sequence of functions defined on $A$ and converging pointwise on $A$ to a function $f$, and if
 (i) $f_n$ is continuous on $A$ for all $n$,
 (ii) $f$ is continuous on $A$,
 (iii) $f_n(x) \geq f_{n+1}(x)$ for all $x \in A$ and all $n$, and
 (iv) $A$ is compact,
 then $\{f_n\}$ converges to $f$ uniformly.
 (b) Show that no three of these conditions are sufficient for uniform convergence. That is, give examples showing that no one of the conditions can be omitted.

**3.** Let $(X, d)$ be a metric space, and let $\mathcal{K}$ be the space of all nonempty compact subsets of $X$. We define the Hausdorff metric $d_H$ on $\mathcal{K}$ as follows: for $A$, $B \in \mathcal{K}$, $d_H(A, B)$ is the smallest $\varepsilon$ such that for every point $a$ in $A$, there exists a point $b$ in $B$ with $d(a, b) \leq \varepsilon$, and for every point $b$ in $B$, there exists a point $a$ in $A$ with $d(a, b) \leq \varepsilon$.

Let $\mathcal{F}_n \subset \mathcal{K}$ be the set of all subsets of $X$ consisting of exactly $n$ points.
 (a) Is $\mathcal{F}_n$ open in $\mathcal{K}$? Closed? Compact?
 (b) Let $f : X \to X$ be a continuous map. Under what conditions will $f$ induce a continuous map from $\mathcal{F}_n$ to $\mathcal{F}_n$?
 (c) Prove or disprove: $\mathcal{K} = \overline{\bigcup_{n=1}^{\infty} \mathcal{F}_n}$ (where $\bar{A}$ represents the closure of $A$).

**4.** Let $f$ be a $C^{n+1}$ function on $\mathbb{R}$, and let $P_n(x)$ be the $n$th-degree Taylor polynomial for $f$ about 0.
 (a) Prove that $|f(x) - P_n(x)| \leq \dfrac{M}{(n+1)!}|x|^{n+1}$, where $M \geq \max |f^{(n+1)}|$ on the interval between 0 and $x$.
 (b) Give an example where equality holds.

**5.** Prove the following theorem: Let $\{f_n : (0, 1) \to \mathbb{R}\}$ be a sequence of differentiable functions converging pointwise to $f : (0, 1) \to \mathbb{R}$. Suppose that the derivatives $\{f_n'\}$ are continuous and converge uniformly to a function $g$. Then $f$ is differentiable and $f' = g$.

**6.** For each of the following, give an example or prove that no such example exists. Denote by $L(f, A)$ the Lebesgue integral of the function $f$ over the set $A$, and by $R(f, A)$ the Riemann integral of the function $f$ over the set $A$.

(a) A function $f$ and a subset $A \subset \mathbb{R}$ such that $L(f, A)$ exists but $R(f, A)$ does not.
(b) A function $g$ and a subset $B \subset \mathbb{R}$ such that $R(f, A)$ exists but $L(f, A)$ does not.
(c) A sequence of functions $\{h_n\}$ converging (in some sense – you choose) to a function $h$ and a subset $C \subset \mathbb{R}$ such that $L(h_n, C) \to L(h, C)$ but $R(h_n, C) \not\to R(h, C)$.

**7.** We say that a function $\mu$ is an integrating factor for the 1-form $\alpha$ if $\mu$ is not identically zero and there exists a function $f$ such that $\mu\alpha = df$. Prove that if $\alpha$ has an integrating factor, then $\alpha \wedge d\alpha = 0$, and show that $\alpha = dz - y\,dx - dy$ has no integrating factor.

**8.** Let $M$ be a nonempty, compact, orientable $n$-dimensional manifold without boundary, and let $\beta$ be an $(n-1)$-form on $M$. Show that $d\beta$ is zero at some point.

**9.** Let $M_2$ be the set of real $2 \times 2$ matrices. We can give it a manifold structure by identifying it with $\mathbb{R}^4$.

(a) Consider the determinant function det. What, if anything, does the Implicit Function Theorem tell us about the behavior of the function near the identity matrix? What about near the matrix with every entry equal to 1?
(b) Same question for the trace function tr.
(c) Is the mapping $F : M_n \to M_n$ given by $F(A) = A^2$ differentiable? If so, what is its derivative?

**10.** Which of the following are manifolds (with or without boundary)?

(a) The set $\{(x, y) : y \leq f(x)\} \subset \mathbb{R}^2$, where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function.
(b) The set $\{(x, y, z) : x^2 + y^2 \leq z^2\} \subset \mathbb{R}^3$.
(c) The set of symmetric, positive definite $2 \times 2$ real matrices. (A matrix is positive definite if all of its eigenvalues are real and positive.)