Do the best that you can to respond to the following problems. It is not required that you respond to every problem, and, in fact, it is much better to provide complete and clearly explained responses to a subset of the given problems than to provide hurried and incomplete responses to all of them. When a complete response is not possible you are encouraged to clearly explain the partial progress that you can achieve.

1. Examples: Provide an example with justification, or explain why no such example exists. It is okay to describe functions using neatly drawn graphs rather than formulas.

   (a) A continuous function $f : \mathbb{R} \to \mathbb{R}$ and a closed set $C \subseteq \mathbb{R}$ such that $f^{-1}(C)$ is open.

   (b) A sequence of bounded functions, (f_n), such that $f_n : [0,1] \to \mathbb{R}$ and $\lim f_n(x)$ does not exist for any $x \in [0,1]$.

   (c) A series $\sum_{n=1}^{\infty} a_n$ where the ratio test indicates convergence, but the root test does not.

2. Let (x_n) be a sequence of real numbers. Show that (x_n) converges if and only if $\sum_{n=1}^{\infty}(x_{n+1} - x_n)$ converges.

3. Consider $\mathbb{R}^N$ with the standard Euclidean norm $||x||_2$. Let $K \subset \mathbb{R}^N$ be compact and let $f : K \to K$ satisfy $||f(x) - f(y)|| < ||x - y||$ for all $x, y \in K$. Show that there is a unique point $x \in K$ such that $f(x) = x$.

4. Let l^{∞} represent the set of all bounded sequences of real numbers and for $(x_n) \in l^{\infty}$ let $||(x_n)||_{\infty} := \sup\{|x_n| : n \in \mathbb{N}\}$ be a norm on l^{∞}. Does l^{∞} possess a countable dense subset?

5. Prove the following version of L'Hospital's Rule: Assume that f and g are continuous on $[0,1]$, differentiable on $(0,1)$, and $f(0) = g(0) = 0$. Show that
$$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{f'(x)}{g'(x)},$$
if the latter limit exists. You may assume standard theorems of one variable calculus.

6. Let $f : [0,1] \to \mathbb{R}$ be continuously differentiable. Show that for all $x, y \in [0,1]$
$$|f(x) - f(y)| \leq \left(\int_0^1 (f')^2 \right)^{\frac{1}{2}} |x-y|^{\frac{1}{2}}.$$
Use this inequality to show that if (f_n) is a sequence of continuously differentiable functions such that $f_n(0) = 1$ for all n and $\int_0^1 (f_n')^2 \leq 1$ for all n, then (f_n) has a subsequence that converges uniformly.

7. Consider the pair of equations
$$\begin{gathered} xy + 2yz = 3xz, \text{ and} \\ xyz + x - y = 1, \end{gathered}$$
for $(x, y, z) \in \mathbb{R}^3$. Observe that $(1, 1, 1)$ is a solution of both equations. What more can you say about solutions of these equations?

8. Suppose that $u(x, y, z)$ is a smooth function that is harmonic in the unit ball in $\mathbb{R}^3$, i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ at all points (x, y, z) where $x^2 + y^2 + z^2 < 1$, and that satisfies $\frac{\partial u}{\partial \nu} = 0$ on the unit sphere, where ν represents the unit outward normal on the sphere and $\frac{\partial u}{\partial \nu}$ represents the directional derivative of u in the direction ν. Show that u is a constant function. (You may not borrow the Maximum Principle from a PDE course unless you intend to prove that result also.)

9. Suppose that $f : [0,1] \to \mathbb{R}$ is continuously differentiable. Show that the set of critical values of f, i.e. $\mathcal{C} := \{f(x) : 0 \leq x \leq 1, f'(x) = 0\}$, has measure zero.