Do the best that you can to respond to the following problems. It is not required that you respond to every problem, and, in fact, it is much better to provide complete and clearly explained responses to a subset of the given problems than to provide hurried and incomplete responses to all of them. When a complete response is not possible you are encouraged to clearly explain the partial progress that you can achieve.

1. Prove that the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has volume $\frac{4}{3}\pi abc$. You may assume that this formula is correct for the spherical case where $a = b = c$.

2. Newton's Method: Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Newton's method is an iterative algorithm designed to approximate the zeros of f. Start with a first guess x_0, then for each integer $n > 0$ find the equation of the tangent line to the graph of f at the point $(x_{n-1}, f(x_{n-1})))$, and solve for the x intercept of that line which is named x_n. Show that the sequence (x_n) is guaranteed to converge under the following conditions:

   (a) f has a zero in the interval (a, b).

   (b) f is twice continuous differentiable with $f'(x) > 0$ and $f''(x) > 0$ on (a, b).

3. A function $f : \mathbb{R}^2 \to \mathbb{R}$ is *Gateux* differentiable at x_0 if the limit

$$\lim_{h \to 0} \frac{f(x_0 + hv) - f(x_0)}{h},$$

   exists for all $v \in \mathbb{R}^2$, i.e. the directional derivative exists in all directions. Either prove the following statement, or find a counterexample: *If f is Gateaux differentiable at the point x_0, then f is continuous at x_0.*

4. The Cauchy Condensation Test claims that if (a_n) is a nonnegative nonincreasing sequence of real numbers whose limit is 0, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} \frac{1}{2^n} a_{2^n}$ converges.

   (a) Use the Cauchy Condensation Test to determine for which $p > 0$ the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.

(b) Is the statement still true if we no longer require (a_n) to be non-increasing? (All other hypotheses remain.)

5. Let X be a metric space with metric d and let $\delta(x,y) := \frac{d(x,y)}{1+d(x,y)} \forall x, y \in X$.

   (a) Show that δ is also a metric on X.

   (b) Show that for any $x_0 \in X$ and any $R > 0$ there is an $r > 0$ such that $\{x \in X : d(x, x_0) < r\} \subseteq \{x \in X : \delta(x, x_0) < R\}$.

   (c) Show that for any $x_0 \in X$ and any $R > 0$ there is an $r > 0$ such that $\{x \in X : \delta(x, x_0) < r\} \subseteq \{x \in X : d(x, x_0) < R\}$.

   You have shown that d and δ are equivalent metrics on X. Even though one is bounded and one is not!

6. Consider the standard Cantor set in $C \subset [0, 1]$.

   (a) Show that C has Lebesgue measure 0.

   (b) Show that $C + C = \{x + y : x, y \in C\} = [0, 2]$.(Hint: If $C_1 := [0, \frac{1}{3}] \bigcup [\frac{2}{3}, 1]$, then $C_1 + C_1 = [0, 2]$.)

7. Is the following version of the Mean Value Theorem true or false when $m > 1$? If true, provide proof. If false, provide a counterexample.

   Theorem 1 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable. Let $a, b \in \mathbb{R}^n$. Then there is a $c \in (a, b)$ such that $f(b) - f(a) = DF(c)(b - a)$. We note that $(a, b) := \{a + t(b - a) : 0 < t < 1\}$.*

8. Show that $f(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases}$ is not Riemann Integrable.

9. Let $F : \mathbb{R}^N \to \mathbb{R}^N : F(x) = Ax + b$ where A is a symmetric N by N matrix with real entries, the eigenvalues of A all satisfy $|\lambda| \leq \frac{1}{2}$, and b is a constant vector in $\mathbb{R}^N$. Let $x_0 \in \mathbb{R}^N$ be arbitrarily chosen, and let $x_n := F(x_{n-1})$. Show that (x_n) converges to a fixed point of F, i.e. a point x such that $F(x) = x$. Also show that this is the unique fixed point. (You are proving a special case of the Contraction Mapping Theorem, so please do not invoke that theorem in your proof.)

10. Provide an example of a sequence of continuous functions on $[0,1]$, $(f_n(x))$, that converge pointwise to a continuous function, $f(x)$, on $[0,1]$, but such that $\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx$. What additional hypothesis beyond pointwise convergence, other than uniform convergence, would guarantee $\lim_{n\to\infty}\int_0^1 f_n(x)dx = \int_0^1 f(x)dx$?

11. Assume the Inverse Function Theorem and prove the Implicit Function Theorem for the case where $f : U \subset \mathbb{R}^3 \to \mathbb{R}^2$ is a continuously differentiable function on the open set U.

12. Suppose that S is a smooth closed surface, such as a sphere or torus, and that F is a smooth vector field. Show that

$$\int\int_S (\nabla \times F) \cdot d\mathbf{S} = 0.$$