Swarthmore College
Honors Exam

Real Analysis and Measure Theory Examination
Spring 1998

Instructions

Do as many problems as you can. Try to give concise proofs showing all of your work. You may quote and use standard results but inadequately supported answers will receive little or no credit.

Part I

Problem 1. Consider the decreasing sequence of nonempty compact sets $A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_n \supset \cdots$ in $\boldsymbol{R}^n$. Prove that the intersection $\cap_{n=1} A_n$ is **compact** and **nonempty**. Is the intersection of every decreasing sequence of nonempty closed sets nonempty?

Problem 2. Prove that the sequence $x, x^2, x^3, \ldots$ converges uniformly on every interval $[0, \epsilon]$ for every $0 < \epsilon < 1$. What can you say about the convergence of the sequence in the interval $[0, 1]$? Can you prove your results by a picture?

Problem 3. Let f_n be a monotone sequence of real valued continuous functions defined on a compact metric space K. Prove or disprove by counterexamples that if the sequence converges to a function f then the convergence is uniform if and only if f is continuous.

Problem 4. Prove that if f is differentiable on $[a, b]$, then for every number s between $f'(a)$ and $f'(b)$ there exists a point $c \in (a, b)$ such that $f'(c) = s$.

Problem 5. Prove that the function

$$f(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0, \end{cases}$$

is differentiable on $\boldsymbol{R}$ but that $f'(x)$ is not continuous at $x = 0$.

Problem 6. Let $f(x) = \displaystyle\int_0^{+\infty} e^{-t^2} \cos(xt)\, dt$.

1. Find the domain and the region of continuity of the function f.
2. Is the function f differentiable everywhere? Using intergation by parts show that $f'(x) = -xf(x)/2$ wherever both sides of the equation make sense.
3. Evaluate $f(x) = \displaystyle\int_0^{+\infty} e^{-t^2} \cos(xt)\, dt$.

Problem 7. Let $f : [a, b] \to \boldsymbol{R}$ be a differentiable function such that $f(a) < 0 < f(b)$ and $0 < m < f'(x) < M$ for all $x \in [a, b]$, then for every $x_0 \in [a, b]$ the sequence defined inductively by

$$x_{n+1} = x_n - \frac{f(x_n)}{M}$$

converges to the unique root $x_* \in [a, b]$ of the equation $f(x) = 0$.

Problem 8. Suppose that the numbers $a_0, a_1, \ldots, a_n$ satisfy

$$a_0 + \frac{a_1}{2} + \cdots \frac{a_n}{n+1} = n\cos(1/n) - n.$$

Prove that the equation

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \sin(x/n) = 0$$

has at least one root between 0 and 1.

Problem 9. Consider the system of equations

$$\sin(x+z) + \ln yz^2 = 0, \qquad e^{x+z} + yz = 0.$$

Note that $(1, 1, -1)$ is a solution of this system.

a. Show that there exists a neighborhood U of the point $(1, 1)$, and a differentiable map $z : U \subset \boldsymbol{R}^2 \to (-1-\epsilon, -1+\epsilon)$ for some $\epsilon > 0$ such that

$$\sin(x + z(x,y)) + \ln yz(x,y)^2 = 0, \qquad e^{x+z(x,y)} + yz(x,y) = 0$$

for all $(x, y) \in U$.

b. Is it possible to find a differentiable function $x = x(y, z)$ defined in a neighborhood W of $(1, -1)$, such that

$$\sin(x(y,z) + z) + \ln yz^2 = 0, \qquad e^{x(y,z)+z} + yz = 0$$

for all $(y, z) \in W$.

Part II

Problem 10. Given a function $f : \boldsymbol{R}^n \to \boldsymbol{R}$. Show that if f is Riemann integrable then so is $|f|$. Is it true that $|f|$ integrable implies that f is integrable? Give a proof or a counterexample. Discuss the same questions for improper Riemann integrals and for Lebesgue integrals. Compare the answers.

Problem 11. Let $I = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a compact interval in $\boldsymbol{R}^n$ and $f : I \to \boldsymbol{R}$ a bounded function which vanishes except on a subset $B \subset I$ of measure zero. Show that the Riemann integral $\int_I f$ exists provided that B is closed. What happens if B is not closed? What can you say about the Lebesgue integrability of f?

Problem 12. Prove that if f is Riemann integrable on the interval (a, b) then it is Lebesgue integrable and the two integrals are equal. Does Lebesgue integrability imply Riemann integrability?

Problem 13. State: Fatou's Lemma, the monotone convergence theorem, and the dominated convergence theorem. Accept as a starting point your favorite of the three results and derive from it the other two.