SWARTHMORE COLLEGE
DEPARTMENT OF MATHEMATICS AND STATISTICS
HONORS EXAMINATION IN REAL ANALYSIS, 2013

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Instructions: Do as many problems or parts of problems or special cases of problems as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are asked to prove.

(1) Let $f : [0,1] \to \mathbb{R}$ be a function. The "graph" of f is the following subset of $\mathbb{R}^2$:

$$G = \{(x, f(x)) \mid x \in [0,1]\}$$

Prove that f is continuous if and only if its graph is compact.

(2) Define $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(x,y) = \sqrt{|xy|}$ (the vertical lines here denote the absolute value). Is f is differentiable at the origin $(0,0)$? Prove your answer.

(3) Among all 4-sided polygons in $\mathbb{R}^2$ that enclose the origin $(0,0)$ and that have perimeter equal to 1, prove that there exists one of maximal area.

(4) Suppose that $K_1 \supset K_2 \supset K_3 \supset \cdots$ are infinitely many nested nonempty compact sets in $\mathbb{R}^n$. Prove that $\cap_{i=1}^{\infty} K_i$ (the intersection of all of these sets) is non-empty. Is the same result true for infinitely many nested nonempty compact sets in a general metric space?

(5) Let X and Y be metric spaces, with Y complete. Let $A \subset X$. Show that if $f : A \to Y$ is uniformly continuous, then f can be uniquely extended to a continuous function from the closure of A to Y, and that this extension is also uniformly continuous.

(6) State the *Implicit Function Theorem* and the *Inverse Function Theorem*. Choose one of these theorems to assume, and show how to prove the other as its corollary.

(7) Let $r_1, r_2, r_3, ...$ be an enumeration of all of the rational numbers which lie in $(0,1)$. Let $\sum_{i=1}^{\infty} c_n$ be a convergent series whose terms are all positive real numbers. Define $f : (0,1) \to \mathbb{R}$ as follows:

$$f(x) = \sum_{\{n \mid r_n < x\}} c_n.$$

In other words, the sum is over all indices n for which $r_n < x$. Prove the following:

(a) f is increasing.
(b) f is continuous at each irrational number in its domain.
(c) f is discontinuous at each rational number in its domain.

(8) Define $F : \mathbb{R}^3 \to \mathbb{R}^4$ as $F(x,y,z) = (x^2 - y^2, xy, xz, yz)$. Let f denote the restriction of F to the unit-sphere $S^2 \subset \mathbb{R}^3$. Prove that the image $f(S^2)$ is a 2-dimensional manifold in $\mathbb{R}^4$. Is this image diffeomorphic to a familiar manifold? Hint: notice that $f(-x,-y,-z) = f(x,y,z)$.

(9) If M is a compact orientable manifold whose boundary ∂M is nonempty, then there does *not* exist a smooth map $f : M \to \partial M$ whose restriction to ∂M equals the identity function. Prove this by providing the details of the following proof sketch: If such a function f exists, then,

$$0 \neq \int_{\partial M} \omega = \int_{\partial M} f^*(\omega) = \int_M df^*(\omega) = \int_M f^*(d\omega) = 0$$

for appropriately chosen ω.

(10) Let $\lambda \in \mathbb{R}$, and define $f : \mathbb{R} \to \mathbb{R}^4$ as $f(t) = (\cos t, \sin t, \cos \lambda t, \sin \lambda t)$. The image, $f(\mathbb{R})$, inherits a topology from $\mathbb{R}^4$. Describe the closure of $f(\mathbb{R})$. Under what condition on λ will f be a homeomorphism onto its image?