SWARTHMORE COLLEGE HONORS EXAM 2012
REAL ANALYSIS

Instructions: Do as many problems, or parts of problems, as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove.

1. Prove or disprove: If $\{p_n\}_{n=1}^{\infty}$ is a sequence of polynomials and $\sum p_n \to f$ uniformly on $\mathbb{R}$ as $n \to \infty$, then f is a polynomial.

2. Let $C = \{f : [0,1] \to [0,1] \mid f \text{ is continuous}\}$, the set of continuous maps from the interval $[0,1]$ to itself. Define a metric d on C by $d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$. Let C_i and C_s be the sets of injective and surjective elements, respectively, of C. Prove or disprove the following:

(a) C_i is closed in C.
(b) C_s is closed in C.
(c) C is connected.
(d) C is compact.

3. Define a sequence of functions $f_1, f_2, \ldots : [0,\infty) \to \mathbb{R}$ by $f_n(x) = \dfrac{\sin(\frac{x}{n})}{x + \frac{1}{n}}$. Discuss the convergence of $\{f_n\}$ and $\{f_n'\}$ as $n \to \infty$.

4. Recall the Intermediate Value Theorem:
Let $f : [a,b] \to \mathbb{R}$ be a continuous function, and y any number between $f(a)$ and $f(b)$ inclusive. Then there exists a point $c \in [a,b]$ with $f(c) = y$.

(a) Prove the Intermediate Value Theorem.
(b) Prove or disprove the following converse to the Intermediate Value Theorem:

If for any two points $a < b$ and any number y between $f(a)$ and $f(b)$ inclusive, there is a point $c \in [a,b]$ such that $f(c) = y$, then f is continuous.

(c) Prove or disprove the following fixed-point theorem:

Let $g : [0,1] \to [0,1]$ be continuous. Then there exists a fixed point $x \in [0,1]$ (that is, a point x such that $g(x) = x$).

5. This question deals with the Riemann integral.

(a) Let S be the unit square $[0,1] \times [0,1]$. Define $f : S \to \mathbb{R}$ by setting

$$f(x,y) = \begin{cases} 1, & \text{if } y \text{ is irrational,} \\ 2x, & \text{if } y \text{ is rational.} \end{cases}$$

For each of the following integrals, compute its value or show that it does not exist:

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy, \quad \int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx, \quad \int_S f(x,y).$$

(b) What conditions on f would guarantee that all three integrals exist and are equal?
(c) Give an example of a function g and a domain D such that $\displaystyle\int_D |g|$ exists but $\displaystyle\int_D g$ does not.

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be smooth (C^∞) and suppose that

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}, \ \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x}.$$

(These are the Cauchy-Riemann equations, which arise naturally in complex analysis.)

(a) Show that $Df(x, y) = 0$ if and only if $Df(x, y)$ is singular, and hence f has a local inverse if $Df(x, y) \neq 0$. Show that the inverse function also satisfies the Cauchy-Riemann equations.

(b) Give an example showing that the statement in part (a) (f has a local inverse if $Df(x, y) \neq 0$) may be false if f does not satisfy the Cauchy-Riemann equations.

7. Let M be a compact 2-manifold in $\mathbb{R}^2$, oriented naturally; give the boundary ∂M the induced orientation. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth (C^∞) function such that $f(\mathbf{x}) = 0$ for any $\mathbf{x} \in \partial M$.

(a) Prove that

$$\int_M f \cdot \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) dx \wedge dy = - \int_M \left(\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right) dx \wedge dy.$$

(b) Deduce from (a) that if, in addition, f is harmonic on M (that is, $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ on M), then $f(\mathbf{x}) = 0$ for any $\mathbf{x} \in M$.

8. (a) (i) Let f be the polar coordinate map given by $(x, y) = f(r, \theta) = (r \cos\theta, r \sin\theta)$. Compute $f^*(dx)$, $f^*(dy)$, and $f^*(dx \wedge dy)$.

(ii) Compute $\int_C xy\, dx$, where $C = \{(x, y) \,|\, x^2 + y^2 = 1, x \geq 0, y \geq 0\}$, the portion of the unit circle in the first quadrant, oriented counter-clockwise.

(b) Let M be a manifold, possibly with boundary. A *retraction* of M onto a subset A is a smooth (C^∞) map $\phi : M \to A$ such that $\phi(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in A$. (For example, the map $\phi(\mathbf{x}) = \mathbf{x}/||\mathbf{x}||$ is a retraction of the punctured plane $\mathbb{R}^2 \backslash \{(0, 0)\}$ onto the unit circle S^1.) Prove the following theorem:

There does not exist a retraction from the plane $\mathbb{R}^2$ onto the unit circle S^1.

(Hint: Consider the 1-form $\dfrac{x\, dy - y\, dx}{x^2 + y^2}$, which is defined in an open set containing S^1.)

9. (a) Let ω_1 and ω_2 be differential forms defined on the same domain.

(i) If ω_1 and ω_2 are closed, must $\omega_1 \wedge \omega_2$ also be closed? If $\omega_1 \wedge \omega_2$ is closed, must ω_1 and ω_2 also be closed?

(ii) If ω_1 and ω_2 are exact, must $\omega_1 \wedge \omega_2$ also be exact? If $\omega_1 \wedge \omega_2$ is exact, must ω_1 and ω_2 also be exact?

(b) Show that every closed 1-form on the punctured space $\mathbb{R}^3 \backslash \{(0, 0, 0)\}$ is exact.