Swarthmore Honors Exam 2011
Real Analysis

Richard A. Wentworth – University of Maryland

Part I : Real Analysis

Instructions. Please answer **three** out of the following four questions. Show all of your work. If you refer to a significant theorem, please state the theorem with all necessary hypotheses. If you answer more than three, please clearly indicate which problems you wish to be graded.

I-1 A topological space is called *separable* if it contains a countable dense subset.

(a) Show that euclidean space $\mathbb{R}^n$ is separable.

(b) Show that every compact metric space is separable.

(c) Let ℓ_∞ be the space of bounded sequences $\mathbf{a} = \{a_j\}_{j=1}^\infty$. We make ℓ_∞ into a metric space by declaring

$$d(\mathbf{a}, \mathbf{b}) = \sup_j |a_j - b_j|$$

Show that ℓ_∞ is not separable.

I-2 Let $\{f_n\}$ be a uniformly bounded sequence of continuous functions on $[a, b]$. Let

$$F_n(x) = \int_a^x f_n(t)dt$$

(a) Show that there is a subsequence of $\{F_n\}$ that converges uniformly on $[a, b]$.

(b) Each F_n is differentiable. Show by an example that the uniform limit in part (a) need not be differentiable.

I-3 Let $\{a_n\}_{n=1}^\infty$ be a positive decreasing sequence $a_n \geq a_{n+1} \geq 0$.

(a) Show that $\sum_{n=1}^\infty a_n$ converges if and only if $\sum_{k=0}^\infty 2^k a_{2^k}$ converges.

(b) Use the result in part (a) to show that the harmonic series $\sum_{n=1}^\infty (1/n)$ diverges.

(c) Use the result in part (a) to show that the series $\sum_{n=2}^\infty (1/n(\log n)^p)$ converges for $p > 1$.

I-4 Let f be a continuous function on the closed interval $[a, b]$.

(a) Show that

$$\lim_{p\to\infty} \left(\int_a^b |f(x)|^p dx \right)^{1/p} = \max_{x\in[a,b]} |f(x)|$$

(b) Give an example of a continuous function f on (a, b) where the improper integrals

$$\int_a^b |f(x)|^p dx$$

exist (i.e. are finite) for all $1 \leq p < \infty$, but

$$\lim_{p \to \infty} \left(\int_a^b |f(x)|^p dx \right)^{1/p} = \infty$$

Part II: Analysis on Manifolds

Instructions. Please answer **three** out of the following five questions. Show all of your work. If you refer to a significant theorem, please state the theorem with all necessary hypotheses. If you answer more than three, please clearly indicate which problems you wish to be graded.

II-1 On $\mathbb{R}^3 - \{0\}$, consider the following 2-form

$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2+y^2+z^2)^{3/2}}$$

(a) Show that ω is closed.

(b) By explicit computation show that

$$\int_{S^2} \omega$$

does not vanish. Here, $S^2 \subset \mathbb{R}^3$ is the unit sphere.

(c) Is ω exact?

II-2 The space $M(n)$ of $n \times n$ matrices is a manifold naturally identified with $\mathbb{R}^{n^2}$. The subspace $S(n)$ of symmetric matrices is a manifold naturally identified with $\mathbb{R}^{n(n+1)/2}$. The space of orthogonal matrices is

$$O(n) = \left\{A \in M(n) : AA^T = I\right\}$$

where T denotes transpose and I is the identity matrix.

(a) Show that $O(n)$ is manifold by considering the map $F : M(n) \to S(n) : A \mapsto AA^T$ (Hint: it might be easier to first consider the derivative of F at the identity, and then use matrix multiplication).

(b) What is the dimension of $O(n)$?

(c) Describe the tangent space of $O(n)$ at the identity matrix as a subspace of $M(n)$.

II-3 Let $A \subset \mathbb{R}^m$, $B \subset \mathbb{R}^n$ be rectangles, $Q = A \times B$, and $f : Q \to \mathbb{R}$ a bounded function.

(a) State necessary and sufficient conditions for the existence of the Riemann integral

$$\int_Q f(x,y)\,dxdy$$

(b) Suppose $\int_Q f(x,y)\,dxdy$ exists. Show that there is a set $E \subset A$ of measure zero such that for all $x \in A - E$, the Riemann integral

$$\int_{\{x\}\times B} f(x,y)\,dy$$

exists.

II-4 Let M be a compact, oriented manifold without boundary of dimension n, and let $S \subset M$ be an oriented submanifold of dimension k. A *Poincaré dual* of S is a closed $(n-k)$-form η_S with the property that for any closed k-form ω on M,

$$\int_M \omega \wedge \eta_S = \int_S \omega$$

(a) Show that if η_S is a Poincaré dual of S, so is $\eta_S + d\alpha$ for any $(n-k-1)$-form α.

(b) Find a Poincaré dual of a point.

(c) Let $S \subset M$ be an embedded circle $S \simeq S^1$ in an oriented 2-dimensional manifold M. Find a Poincaré dual to S (Hint: use the fact that there is a neighborhood of S in M of the form $S^1 \times (-1,1)$).

II-5 Let M be a differentiable manifold. If X is a smooth tangent vector field on M, then X gives a map $C^\infty(M) \to C^\infty(M)$ on smooth functions by $f \mapsto X(f) = df(X)$. Explicitly, in local coordinates $x_1, \ldots, x_n$,

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x_i} \quad , \quad X(f) = \sum_{i=1}^n X^i \frac{\partial f}{\partial x_i}$$

(a) Show that if X_1 and X_2 are vector fields such that $X_1(f) = X_2(f)$ for all $f \in C^\infty(M)$, then $X_1 = X_2$.

(b) If X, Y are vector fields on M, define the *Lie bracket* $[X,Y]$ to be the vector field determined by the condition that

$$[X,Y](f) = X(Y(f)) - Y(X(f))$$

for all $f \in C^\infty(M)$. Compute $[X,Y]$ in local coordinates.

(c) Show that if $N \subset M$ is a smooth submanifold of M and X, Y are two tangent vector fields to N, then $[X,Y]$ is also tangent to N.