Honors Exam in Real Analysis and Real Analysis II
Swarthmore College
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Attempt at least 3 questions from Part I and at least 3 questions from Part II. You need not prove standard results or state standard definitions to use them, but do invoke them explicitly enough that your arguments are clear (e.g. "By the definition of connected, we know that . . . ").

Part I — Real Analysis

I-1. If we equip the set E of all continuous functions $f : [0, 1] \to \mathbb{R}$ with the metric

$$d(f, g) = \max_{x\in[0,1]} |f(x) - g(x)|,$$

E becomes a complete metric space. Define

$$S = \{f \in E \ : \ f'(x) \text{ exists for all } x \in (0, 1)\ \}.$$

Prove that S is neither open nor closed in E.

I-2. Let $\mathcal{K}$ denote the set of nonempty compact subsets of $\mathbb{R}^2$, and write d for the Euclidean metric on $\mathbb{R}^2$. Given $A \in \mathcal{K}$ and $r \geq 0$, write

$$A_r = \{x \in \mathbb{R}^2 \ : \ d(x, y) \leq r \text{ for some } y \in A\ \}.$$

Observe that $A_0 = A$, and that $A_s \subset A_t$ whenver $s \leq t$.

a) Given $A, B \in \mathcal{K}$, define

$$\mu(A, B) = \min\{r \geq 0 \ : \ B \subset A_r\ \}.$$

Show that this definition makes sense — that is, that the above minimum in fact exists.

b) The *Hausdorff metric on* $\mathcal{K}$ is defined as

$$\rho(A, B) = \max(\mu(A, B), \mu(B, A)).$$

Prove that ρ is in fact a metric.

c) Show, with an explicit example, that $\mu : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ is *not* a metric. (Hint: Consider the case $B \subset A$.)

I-3. Let E be the space $[0, 1]$ equipped with the *discrete metric*:

$$d(x, y) = \begin{cases} 0, & x = y; \\ 1, & x \neq y. \end{cases}$$

a) Show that any function $f : E \to \mathbb{R}$, where $\mathbb{R}$ has its usual metric, is continuous.

b) Is E connected?

c) Is E compact?

I-4. a) Prove that

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0.$$

b) Does

$$\sum_{k=1}^{\infty} \frac{k!}{k^k}$$

converge?

I-5. Given $a > 0$ and two continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, let us define the function $h : [0, a] \to \mathbb{R}$ by the formula

$$h(x) = \int_{g(0)}^{g(x)} f(s)\ ds.$$

a) Show that h is continuous on $[0, a]$.

b) Show via an explicit example that h need not be differentiable on $(0, a)$. (Hint: try making f a constant function.)

c) Can you impose an additional condition on g or on f that guarantees that $h(x)$ *is* differentiable on $(0, a)$? (Simple and/or highly restrictive conditions are OK!)

Part II — Real Analysis II

II-1. In this problem, all matrices and vectors are given with respect to the standard basis.

a) Find the dimension of $\mathcal{A}^2(\mathbb{R}^4)$, the space of alternating 2-tensors on $\mathbb{R}^4$.

b) Verify that the following map $f : (\mathbb{R}^4)^2 \to \mathbb{R}$ is a member of $\mathcal{A}^2(\mathbb{R}^4)$.

$$f\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}\right) = (x_1 + x_3)y_2 - x_2(y_1 + y_3).$$

c) Compute

$$T^* f\left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right),$$

where f is as above and $T : \mathbb{R}^2 \to \mathbb{R}^4$ is the linear transformation given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 3 \\ 0 & 0 \end{pmatrix}.$$

II-2. Let $U \subset \mathbb{R}^{n-1}$ be an open set, and let $\alpha : U \to \mathbb{R}^n$ be a smooth coordinate patch (in particular, α has Jacobian of rank $n-1$ everywhere). Let $g : \mathbb{R}^n \to \mathbb{R}$ be a smooth function such that $g(y) = 0$ and $Dg(y) \neq 0$ for all $y \in \alpha(U)$.

a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, and supppose that the restriction of f to $\alpha(U)$ has a strict maximum: that is, there is some $p \in \alpha(U)$ such that

$$f(p) > f(q) \text{ for all } q \in \alpha(U) \setminus \{p\}.$$

Prove that $Df(p) = \lambda Dg(p)$ for some real number λ.

b) Use part a) to find the maximum of the function $f(x, y, z) = x + y + z$ on the set $\alpha(U)$, where

$$U = \{(x, y) \ : \ 0 < x < 1 \text{ and } 0 < y < 1\ \}$$

and

$$\alpha(x, y) = (x, y, 6x - 6x^2 + 3y - 3y^2).$$

(Hint: Take $g(z, x, y) = z - 6x + 6x^2 - 3y + 3y^2$.)

[Remark: In this situation, one could of course just optimize $f \circ \alpha$ directly on U — indeed, you should do this to check your work! In applications where the "constraint surface" $\{\ g = 0\ \}$ is not given explicitly as the graph of some known α, though, the method outlined in this problem can be useful. The method is called the *method of Lagrange multipliers*.]

II-3. In multivariable calculus, we learn about integration with polar coordinates. Here is a particular case: Suppose that U is an open bounded region in the x-y plane that is easily described in polar coordinates by $a \leq r \leq b$ and $c \leq \theta \leq d$ — that is,

$$U = \{\ (r\cos(\theta), r\sin(\theta)),\ \ r \in (a,b) \text{ and } \theta \in (c,d)\ \},$$

where $(a,b) \subset (0,\infty)$ and $(c,d) \subset (0,2\pi)$. Suppose that $f : U \to \mathbb{R}$ is a smooth function. Then we have that

$$\int_U f(x,y) = \int_c^d \int_a^b f(r\cos(\theta), r\sin(\theta))r\ dr\ d\theta.$$

Give a rigorous justification for this formula.

II-4. Let S^2 be the unit sphere in $\mathbb{R}^3$. Compute

$$\int_{S^2} xz\ dx \wedge\ dy + 3y\ dx \wedge dz + y\ dy \wedge\ dz.$$

II-5. Let $[a,b]$ be an interval in $\mathbb{R}$, and let $\alpha : [a,b] \to \mathbb{R}^2$ be a smooth coordinate patch (i.e. α is 1-to-1, smooth on (a,b), has nonzero Jacobian everywhere on (a,b), and has continuous inverse on its image). Let us write $\gamma = \alpha((a,b))$ for the image of α in $\mathbb{R}^2$.

In advanced calculus we define the *length of* γ in two ways. One is the "surface measure" formula

$$\int_\gamma 1\ dS = \int_a^b \det M(t),$$

where $M(t)$ is the 2×2 matrix whose first column is $D\alpha(t)$ (the Jacobian of α at t) and whose second column is the unit normal to γ at $\alpha(t)$, chosen so that $\det(M(t)) > 0$.

The more common definition of the length of γ (as given, say, in Munkres' *Analysis on Manifolds*) is the "one-dimensional volume" formula

$$\int_\gamma 1\ dV = \int_a^b V(D\alpha(t)),$$

where

$$V(D(\alpha(t)) = \sqrt{\det\left[D\alpha(t)^{tr} D\alpha(t)\right]}.$$

a) Show that these two definitions of the "length of γ" actually agree.

b) Rewrite the integral $\int_\gamma 1\ dS$ as the integral of a 1-form on γ.