2009 Honors Examination: Real Analysis

I. Basic Analysis

1. (a) Define the terms "compact" and "complete," as they apply to a metric space.

(b) Show that in a compact metric space, any sequence has a convergent subsequence.

(c) Show that if a Cauchy sequence has a convergent subsequence then the sequence itself is convergent.

(d) Show that any compact metric space is complete.

2. (a) Prove that the sum $\displaystyle\sum_{n=0}^{\infty} \frac{x^n \cos(nx)}{n!}$ converges for all $x \in \mathbb{R}$. Prove that it does not converge uniformly on the real line, but nevertheless defines a continuous function on $\mathbb{R}$. State precisely any theorems you use.

(b) How about the sum of the derivatives? Does it converge pointwise? Does it converge uniformly on $\mathbb{R}$? Does it converge to the derivative of the function studied in **(a)**? Once again, state precisely any theorems you use.

3. Let $f(t)$ be the "inch-long ruler function" defined on the interval $[0, 1]$ by

$$f(t) = \begin{cases} 2^{-n} & \text{if } t = 2^{-n}k, \text{ where } n, k \in \mathbb{Z} \text{ and } k \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$$

(a) Where is $f(t)$ continuous?

(b) Is $f(t)$ Riemann integrable? If so, what is its integral?

II. Analysis on Manifolds

4. (a) Let A be a subset of $\mathbb{R}^n$. Define what is meant by "A is of measure zero," "the boundary of A," and "the closure of A."

(b) If A has measure zero, must its closure have measure zero? Must its boundary have measure zero? Provide proofs or counterexamples.

5. For what values of a, b is the surface defined by $w^2 + x^2 + y^2 + z^2 = a$ and $wxyz = b$ a submanifold of $\mathbb{R}^4$? State precisely any theorems you use in proving your claim. Give an equation of the tangent plane at $(1, 2, 3, 4)$ to the surface in this family which contains that point.

6. **(a)** Give an example of a smooth 2-form ω_0 on $\mathbb{R}^3 - \{0\}$ (where 0 is the origin) which is closed (i.e. $d\omega_0 = 0$) but not exact. (Hint: in terms of vector fields, you might want to consider a vector field of the form $\vec{r}/r^p$ for some p.)

(b) Let $a, b \in \mathbb{R}$ and let p, q be distinct points in $\mathbb{R}^3$. Use the form ω_0 to define a smooth 2-form ω on $\mathbb{R}^3 - \{p, q\}$ which is closed but has the property that if S_p is a sphere centered at p with radius $||p - q||/2$, S_q is defined analogously, and both are oriented so that the "positive" side is out, then

$$\int_{S_p} \omega = a\,, \qquad \int_{S_q} \omega = b\,.$$

If I give you a smooth closed surface S in $\mathbb{R}^3$ such that $p \notin S$ and $q \notin S$, what do you have to know in order to compute $\displaystyle\int_S \omega$?

7. Let H be the upper hemisphere in $\mathbb{R}^3$,

$$H = \{(x, y, z) : x^2 + y^2 + z^2 = 1, z \geq 0\}\,,$$

with normal vector pointed outward. Compute

$$\int_H z\,dx \wedge dy\,.$$