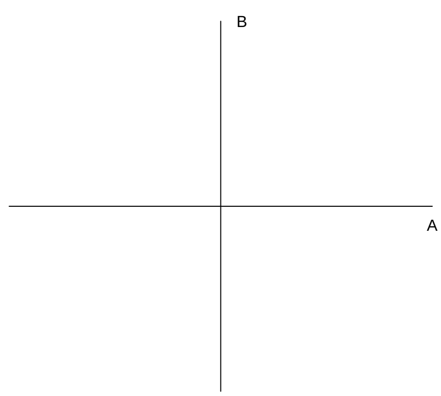
Honors Exam in Real Analysis
Swarthmore College
Spring 2008

Answer as many questions as you can, taking care to answer questions from both parts I and II. If you need to choose between answering most questions carefully and all questions hurriedly, answer most questions carefully. You may quote any standard result as long at it is not essentially what I am asking you to prove.

Part I — Real Analysis

1. Find a metric $d(\cdot,\cdot)$ on the real line $\mathbf{R}$ that makes $\mathbf{R}$ into a bounded set: i.e. there is some $M > 0$ such that $d(x,y) \leq M$ for any $x, y \in \mathbf{R}$. Verify that d is in fact a metric.

2. Let A be the line segment $\{ (x,0) \ : \ -1 < x < 1 \}$ and let B be the line segment $\{ (0,y) \ : \ -1 < y < 1 \}$. Suppose that $f : A \cup B \to A$ is a continuous function (view $A \cup B$ and A as subsets of the plane, with its usual metric). Prove that f cannot be one-to-one.



3. Suppose that X is a subset of a metric space E. Define the following number:

$$\alpha(X) = \inf(\delta \ : \ X \text{ can be covered by finitely many open balls in } E \text{ of radius } \delta \ \}.$$

a) Show that, if X is compact, then $\alpha(X) = 0$.

b) Show that, if X is not closed, X need not be compact even if $\alpha(X) = 0$.

(If E is complete and X is closed, then $\alpha(X) = 0$ if and only if X is compact. This is a version of the so-called *Bolzano-Weierstrass theorem*. The function $\alpha(X)$ is called a *measure of noncompactness*.)

4. If we equip the set E of all continuous functions $f : [0,1] \to \mathbf{R}$ with the metric

$$d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|,$$

E becomes a complete metric space. Let X be the closed unit ball in this space:

$$X = \{ \text{ continuous functions } f : [0,1] \to \mathbf{R} \text{ such that } |f(x)| \leq 1 \text{ for all } x \in [0,1] \ \}.$$

Show that $\alpha(X) = 1$, where α is as in problem 3. Conclude that X, while closed and bounded, is *not* compact.

5. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a continuous function. Choose and fix some closed interval $[a,b]$. For each $k \in \mathbf{N}$, define the function

$$g_k(x) = k \int_{x-(1/k)}^{x} f(t) \ dt.$$

Show that each $g_k(x)$ is differentiable for all x, and that $g_k \to f$ uniformly on $[a,b]$ as $k \to \infty$. You may use the fundamental theorem of calculus.

(This problem shows that a continuous function can be uniformly approximated by differentiable functions on any closed interval.)

Part II — Real Analysis II

6. The picture below shows the boundary of the image R of the open triangle

$$U = \{\ (x,y)\ :\ x \in (0,1) \text{ and } y \in (0,x)\ \}$$

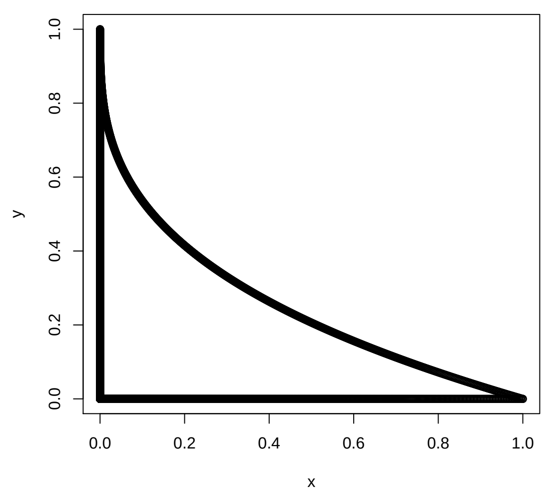
under the map

$$T(x,y) = \big((x-y)^3, xy\big)\,.$$

a) Show that T is one-to-one.
b) Show that R is open.
c) Integrate the function

$$f(u,v) = \sqrt{\frac{1}{u^{2/3} + 4v}}$$

over R.
d) Compute the area of R. (Hint: there are several ways to do this, including the various ways to integrate the constant function 1 over R.)



7. A smooth function $u : \mathbf{R}^2 \to \mathbf{R}$ is called *harmonic* if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

everywhere in $\mathbf{R}^2$.

Suppose that $u : \mathbf{R}^2 \to \mathbf{R}$ is harmonic, let C be the unit circle about the origin, and let ω be the 1-form

$$\omega(x, y) = \frac{\partial u}{\partial x}\, dy - \frac{\partial u}{\partial y}\, dx.$$

a) Compute $d\omega$.

b) Use either Green's or Stokes' theorem, together with the fact that u is harmonic, to show that

$$\int_C \omega = 0.$$

c) Use the parameterization

$$p : [0, 2\pi] \to \mathbf{R}^2 \ : \ p(t) = (\cos(t), \sin(t))$$

of the unit circle C to write

$$\int_C \omega$$

explicity as an integral in t from 0 to 2π.

d) Write the directional derivative of u at the point $(\cos(t), \sin(t))$ in the direction $(\cos(t), \sin(t))$ (that is, in the direction of the outer normal to the unit circle). Conclude that, if u is harmonic, its "average" directional derivative normal to the unit circle is zero.

8. Let $\{f_k\}$ be a sequence of measurable functions on $[0,1]$, where $[0,1]$ is equipped with Lebesgue measure m. We say that f_k *converges to* f *in* $L^1[0,1]$ if and only if

$$\lim_{k\to\infty} \int_{[0,1]} |f_k - f| \, dm = 0.$$

On the other hand, we say that f_k *converges to* f *in measure on* $[0,1]$ if

$$m(\{x \in [0,1] \ : \ |f_k(x) - f(x)| \geq \epsilon\}) \to 0 \text{ as } k \to \infty$$

for any $\epsilon > 0$. Prove that convergence in $L^1[0,1]$ implies convergence in measure, but not conversely.

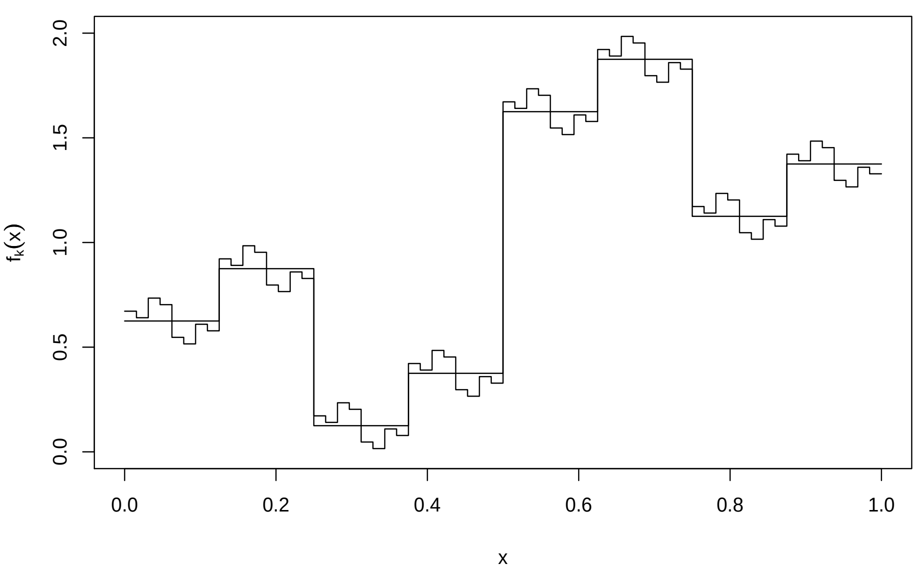
9. Suppose that the sequence g_k of functions on $[0,1]$ is defined by

$$g_k(x) = \frac{(-1)^k}{2^k}(-1)^{\text{Floor}(2^k x)},$$

where Floor(y) is just the greatest integer less than or equal to y. Define, for each integer $m \geq 0$,

$$f_m(x) = \sum_{k=0}^{m} g_k(x).$$

The functions $f_3(x)$ and $f_6(x)$ are pictured below.



Prove that the pointwise limit $f = \lim_{m\to\infty} f_m$ exists and that it is integrable, and find its integral.