Swarthmore College
Honors Exam in Real Analysis

May 2007

Instructions: Do as many of the following problems as you can. Justify all answers. You may quote any standard result as long as that result is not essentially what you are being asked to prove. If you do quote a standard result, make sure you clearly identify the result and verify that the hypotheses are satisfied.

In these problems, a function f is said to be of class C^k if f is continuous and all of its (ordinary or partial) derivatives up through order k exist and are continuous. It is said to be *smooth* (or of class C^∞) if it is of class C^k for every $k \geq 0$.

1. Suppose K is a subset of $\mathbb{R}^n$. Prove that K is compact if and only if every continuous function $f\colon K \to \mathbb{R}$ is bounded.

2. Suppose $f\colon (0,1] \to \mathbb{R}$ is differentiable and satisfies $|f'(x)| < 1$ there. Show that the sequence $\{f(1/n)\}$ converges.

3. Suppose $f\colon [0,\infty) \to \mathbb{R}$ is of class C^2, and $f(x) \to 0$ as $x \to \infty$.
   (a) If $f'(x) \to b$ as $x \to \infty$, show that $b = 0$.
   (b) If f'' is bounded, show that $f'(x) \to 0$ as $x \to \infty$.
   (c) Give an example of such an f for which $f'(x)$ does not converge as $x \to \infty$.

4. Suppose $f\colon [0,1] \to \mathbb{R}$ is *upper semicontinuous:* This means that for every $x \in [0,1]$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that $|y - x| < \delta$ implies $f(y) < f(x) + \varepsilon$. Prove that f is bounded above and achieves its maximum value at some $x \in [0,1]$.

5. Define a function $f\colon [0,1] \to \mathbb{R}$ by setting $f(x) = 1/n$ if the first 0 after the decimal point in the decimal expansion of f occurs in the nth place after the decimal point, and $f(x) = 0$ if there are no 0's after the decimal point. Prove that f is Riemann-integrable. (To avoid ambiguity, choose decimal representations ending in 9's instead of 0's when both are possible. For example,
$$\begin{aligned} f(1/10) &= f(0.0999\ldots) = 1,\\ f(1/2) &= f(0.4999\ldots) = 0,\\ f(1) &= f(0.9999\ldots) = 0. \end{aligned}$$
Note that only digits after the decimal point are counted.)

6. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^k$ is continuous. Let λ be a positive real number, and assume that for every $x \in \mathbb{R}^n$ and $a > 0$, $f(ax) = a^\lambda f(x)$.
   (a) If $\lambda > 1$, show that f is differentiable at 0.
   (b) If $0 < \lambda < 1$, show that f is not differentiable at 0.
   (c) If $\lambda = 1$, show that f is differentiable at 0 if and only if it is linear.
7. Let M_2 be the set of 2×2 real matrices, identified with $\mathbb{R}^4$ in the obvious way, and define $F\colon M_2 \to M_2$ by $F(X) = X^2$ (i.e., the matrix X multiplied by itself). Does F have a local smooth inverse in a neighborhood of I (the 2×2 identity matrix)? Answer the same question when I is replaced by
$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
   Prove your answers correct.
8. Let $T \subset \mathbb{R}^3$ denote the doughnut-shaped surface obtained by revolving the circle $(y-2)^2 + z^2 = 1$ around the z-axis. Give T the orientation determined by the outward unit normal.
   (a) Compute the surface area of T.
   (b) Compute the integral $\int_T \omega$, where ω is the 2-form $z\,dx \wedge dy$.
9. Suppose M is a smooth, compact n-manifold with boundary in $\mathbb{R}^n$. If f is a smooth real-valued function and X is a smooth vector field on $\mathbb{R}^n$, use the general version of Stokes's theorem to prove the following "integration by parts formula":
$$\int_M \langle \operatorname{grad} f, X\rangle\, dV = \int_{\partial M} f\langle X, N\rangle\, dV - \int_M f \operatorname{div} X\, dV,$$
   where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product or dot product, and N denotes the outward unit normal to ∂M. Explain what this has to do with integration by parts.