Honors Exam in Real Analysis I and II

Please answer as many questions or parts of questions as you can. You may quote and use standard results, as long as they are fully explained and justified. You may also use the statement of any part of a question in any subsequent part. Show all work. Good luck!

1. (a) Let $\{a_n\}$ be a sequence of nonnegative numbers and suppose that $\lim_{n\to\infty} a_n = 0$ and $\sum a_n = \infty$. Show that for every $0 < p < q$, there is a finite set of terms $\{a_{n_1}, \ldots a_{n_k}\}$ with $p < \sum_{i=0}^{k} a_{n_i} < q$.

   (b) A series diverges unconditionally to infinity if every rearrangement of the terms diverges to ∞. Show that a series $\sum_{i=0}^{\infty} x_i$ diverges unconditionally to ∞ if and only if the sum of the positive terms of $\{x_i\}$ diverges to ∞ and the sum of the negative terms of $\{x_i\}$ converges.

2. (a) Let C^1 be the space of all real-valued functions on $[0, 1]$ for which f' is continuous on $[0, 1]$. Show that for any function $f \in C^1$, $\lim_{n\to\infty} \int_0^1 f(x)\sin(2\pi nx)dx = 0$. (Hint: integrate by parts.)

   (b) Show that if $f_n : [0, 1] \to \mathbb{R}$ are integrable and converge uniformly to some $f \in C^1$, then $\lim_{n\to\infty} \int_0^1 f_n(x)\sin(2\pi nx)dx = 0$.

3. Let $\mathcal{P}$ be the set of polynomials defined on $\mathbb{R}$.

   (a) Show that for any function $f(x) = \sum_{i=0}^{\infty} a_i x^i$, where the power series converges absolutely on $\mathbb{R}$, there is a sequence of polynomials $p_1, p_2, \ldots \in \mathcal{P}$ such that $\lim_{n\to\infty} p_n(x) = f(x)$ for every $x \in \mathbb{R}$.

   (b) What is the set of all functions f for which there exist $p_n \in \mathcal{P}$ such that the sequence $\{p_n\}$ converges uniformly to f on $\mathbb{R}$? Prove your answer is correct.

4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on all of $\mathbb{R}$ and assume that $f'(x) \neq 1$ for any x. Prove that there is at most one point $t \in \mathbb{R}$ for which $f(t) = t$.

   (b) We say that $f(x) = O(g(x))$ as $x \to \infty$ if there is are constants $c, M > 0$ with $f(x) \leq cg(x)$ whenever $x \geq M$. Suppose that a real-valued function f is differentiable on $(0, \infty)$ and that $f'(x) = O(x)$ as $x \to \infty$. Prove that $f(x) = O(x^2)$ as $x \to \infty$.

5. Suppose that a function $f : U \to \mathbb{R}$ is n times differentiable on some open interval U containing x_0. Assume that $f^{(n)}$ is continuous on U and that n is the smallest positive integer with $f^{(n)}(x_0) \neq 0$. Prove that if n is odd then x_0 is a not a local minimum or a local maximum for f.

6. Let (X, ρ) and (Y, σ) be metric spaces. Define $X \times Y := \{(x, y) : x \in X, y \in Y\}$ and $\tau((x_1, y_1), (x_2, y_2)) := \rho(x_1, x_2) + \sigma(y_1, y_2)$.

   (a) Prove that $(X \times Y, \tau)$ is a metric space.

   (b) Let (X, ρ) be compact. If $f : X \to Y$, prove that f is continuous if and only if the set $G_f = \{(x, f(x)) : x \in X\}$ is compact in $X \times Y$. G_f is called the graph of f.

   (c) Give an example of a map $f : \mathbb{R} \to \mathbb{R}$ such that G_f is closed in $\mathbb{R}^2$, but f is not continuous.

7. Define $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$. (This is the formula for converting spherical coordinates to rectangular coordinates.) Is f locally invertible for every point in $\mathbb{R}^3$? If not, show where it is not locally invertible. If it is locally invertible everwhere, is it globally one-to-one? Justify your answers.

8. Let $\omega = xdx + ydy + zdz$, a one-form.

   (a) Show that in the spherical coordinates (ρ, θ, ϕ) given in # 7, with $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta, z = \rho \cos\phi$, we have $\omega = \rho d\rho$.

   (b) Let γ be the path from $(0, 0, 0)$ to $(1, 1, 1)$ shown below. Calculate $\int_\gamma \omega$. (You should not need the formula for γ.)