Real Analysis, Analysis on Manifolds, and Measure Theory Examination
Swarthmore College
Honors Exam

Spring 2001

Instructions: Please do as many problems as you can. Try to give concise proofs showing all of your work. You may quote and use standard results, but you need to fully explain your reasons. Of course, if you are proving a standard result, you may not quote that result as the proof. Inadequately supported answers will receive little or no credit. Good luck!

Notation: The set of natural numbers is denoted $\mathbb{N}$; the set of integers is denoted $\mathbb{Z}$, the set of rationals is denoted $\mathbb{Q}$ and the set of reals is $\mathbb{R}$. If M and N are metric spaces, $\mathcal{C}(M, N)$ will be the set of continuous functions from M to N. A function is smooth or C^∞ on an open set if it has derivatives to all orders and they are continuous on the set.

1. (a) Prove using the Least Upper Bound Property for $\mathbb{R}$ that $\mathbb{N}$ is not bounded above.
   (b) Does $\mathbb{Q}$ satisfy the Least Upper Bound Property? Either prove it does or provide specific counterexample.

2. Let M and N be metric spaces and assume M is compact.
   (a) Let $f : M \to N$ be continuous. Prove that $f(M) = \{f(x) \mid x \in M\}$ is compact in N.
   (b) If $f : M \to N$ is not continuous is $f(M)$ necessarily compact? If so prove it. If not, give a specific counterexample (specific M, N, and f) and show it is a counterexample.

3. Describe all continuous functions from $\mathbb{R}$ to the set of integers, $\mathbb{Z}$, and prove your result.

4. Prove using the definition of uniform convergence (using ϵ and N) that the function sequence $\{f_k \mid k = 1, 2, \ldots\}$, where

$$f_k(x) = \frac{2kx^3 + x^5 + 3}{2k + x^2},$$

   converges uniformly on $\mathbb{R}$.

5. Let $f : [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$ and differentiable on $(0, 1)$. Assume also $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all $x \in (0, 1)$. Prove $f(x) = 0$ for all $x \in [0, 1]$.

6. Let $f(x) = \displaystyle\sum_{n=1}^{\infty} \frac{n \sin nx}{n^4 + 1}$.
   (a) Where does f converge pointwise? Where does f converge uniformly? Where does f converge uniformly absolutely? Why?
   (b) What is f'? Explain why the series for f' converges and is equal to f'.

7. Prove that the map $F : \mathcal{C}([0, 1], \mathbb{R}) \to \mathcal{C}([0, 1], \mathbb{R})$ defined by

$$F(f)(x) = 1 + \int_0^x \frac{t}{4} f(t)dt$$

   is a contraction mapping. Write out the first four successive approximations (up to p_4) starting with $p_0 = 0$. Please try to find the pattern in p_n and try to find the fixed point P as an infinite sum and as a function in closed form. How could you check that your solution P is the fixed point?

8. Let M be a compact subset of $\mathbb{R}^n$ and let $k \in \{1, 2, \ldots, n-1\}$.

   (a) Give the definition that M is a $k-$dimensional smooth submanifold of $\mathbb{R}^n$.

   (b) Let M be an $(n-1)-$dimensional submanifold of $\mathbb{R}^n$ and let $f : M \to \mathbb{R}$ be continuous. Explain how to integrate f on M.

   (c) Find $\int_{S^2}(x^2 + y^2)dV$ where S^2 is the unit sphere in $\mathbb{R}^3$ and dV is the standard volume element.

9. Quote the Inverse Function Theorem and use it to explain why the system

$$\begin{aligned} u(x, y, z) &= x + xyz \\ v(x, y, z) &= y + xy \\ w(x, y, z) &= z + zx + 3z^2 \end{aligned}$$

   can be solved for (x, y, z) in terms of (u, v, w) near $(0, 0, 0)$.

10. Let $f(x_1, x_2, x_3, x_4) = x_1^2+2x_2^2+x_3^2+4x_4^2$ and let $S = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,|\, f(x_1, x_2, x_3, x_4) = 1\}$. Show that S is a 3-manifold.

11. Compare and contrast Riemann and Lebesgue integration on $[a, b]$ by answering the following questions.

    (a) Give a condition involving sets of discontinuity that is equivalent to Riemann integrability (Lebesgue's Theorem). Apply this result to explain why a bounded piecewise continuous function on $[a, b]$ is Riemann integrable.

    (b) Let $\phi : [a, b] \to \mathbb{R}$ be a simple function. What does that mean? What is the Lebesgue integral, $\int_{[a,b]} \phi$?

    (c) Let $f : [a, b] \to \mathbb{R}$ be bounded. What must f satisfy to be Lebesgue integrable by the definition. What is the definition of $\int_{[a,b]} f$?

    (d) Give an example of a bounded function $f : [0, 1] \to \mathbb{R}$ that is Lebesgue integrable but not Riemann integrable and prove f is not Riemann integrable. Explain why f is Lebesgue integrable.

12. Let $f_n : [0, 1] \to \mathbb{R}$ be Lebesgue measurable for $n \in \mathbb{N}$, and let $f : [0, 1] \to \mathbb{R}$ be Lebesgue measurable.

    (a) Assume $f_n \to f$ uniformly on $[0, 1]$. Prove $\int_{[0,1]} f_n \to \int_{[0,1]} f$.

    (b) Assume $f_n \to f$ pointwise on $[0, 1]$. Does $\int_{[0,1]} f_n \to \int_{[0,1]} f$? Either prove this statement or give a counterexample.