SWARTHMORE COLLEGE
Department of Mathematics and Statistics
Honors Examination

9 May 1996
1:30–4:30

Real Analysis

DIRECTIONS: Do as many problems, or parts of problems, as you can.

1. Prove, disprove or give a counterexample to each of the following:
   a) Every closed and bounded subset of a complete metric space is compact.
   b) Let $\{f_n\}$ be a sequence of Riemann-integrable functions on $[a, b]$ and define
   $$F_n(x) = \int_a^x f_n(t)\,dt$$
   for $x \in [a, b]$. Then there exists a subsequence $\{F_{n_k}\}$ that converges uniformly on $[a, b]$.
   c) Suppose that I is a bounded interval and that $f_n: I \to \mathbf{R}$ is a sequence of continuous functions.
      1) If f_n converges uniformly to 0, then $\int_I f_n(x)\,dx \to 0$.
      2) If f_n converges pointwise to 0, then $\int_I f_n(x)\,dx \to 0$.
   d) A uniform limit of differentiable functions on $[0, 1]$ is differentiable at at least one point.

2. With each step justified, compute
$$\lim_{a\to\infty} \lim_{n\to\infty} \int_0^a x(1 - \frac{x}{n})^n\, e^{x/2}\, dx.$$

3. Suppose that $X \subset \mathbf{R}$ is uncountable. Prove that there exists some $x_0 \in \mathbf{R}$ so that every open interval I containing x_0 satisfies $I \cap X \neq \emptyset$.

4. A subset Y in a metric space X is called a G_δ-set if it is the countable intersection of open sets. Given $f: X \to \mathbf{R}$, show that $A = \{x \in X \mid f \text{ is continuous at } x\}$ is a G_δ-set in X.

5. Prove that the series
$$\sum_{n=1}^{\infty} n \sin(x^n)$$
converges absolutely on $[-\delta, \delta]$ for all $\delta \in (0, 1)$.

6. Suppose f is a continuously differentiable map of $I = [0, 1]$ to itself with $f(0) = f(1) = 0$. Denote by $f^{\circ n}$ the n-fold composition of f with itself. We say that $x_0 \in (0, 1)$ is a *local attractor* for f if there is an open interval about x_0 on which $f^{\circ n}(x) \to x_0$ for all x in the interval.
   a) State and prove a condition guaranteeing that x_0 is a local attractor,
   b) Interpret the condition of being a local attractor in terms of the graph of f.
   c) Is there a higher dimensional analog to your condition?

7. Suppose that $f(x, y, z) = x$ and that R is the unit cube defined by $0 \leq x, y, z \leq 1$ in $\mathbf{R}^3$. We wish to integrate f on R. Do it three ways.

   a) Use iterated integration.
   b) Use only the definition of a volume integral.
   c) Use Stokes' Theorem.

8. Suppose that $f: \mathbf{R}^n \to \mathbf{R}^n$ is differentiable with derivative Df.

   a) What can you conclude if Df is zero everywhere?
   b) What can you conclude if Df is an orthogonal transformation (i.e., represented by an orthogonal matrix)?
   c) Suppose that F is a vector field defined on a connected open set D in $\mathbf{R}^2$. Under what conditions is there a function Φ so that $F = \operatorname{grad} \Phi$?

9. Suppose $f: D \to \mathbf{R}^2$ where $D \subset \mathbf{R}^2$ and f is continuously differentiable.

   a) Show that f maps sets of measure zero to sets of measure zero.
   b) Fix $x_0 \in D$ and let $A = Df(x_0)$. Assume that $\|Au\|$ is not independent of the choice of unit vector u (where $\|Au\|$ denotes the length of Au).
      1) Show that there exists a unit vector u_1 (respectively, u_2), unique up to sign, so that $\|Au_1\|$ is maximal (respectively, minimal) over all choices of unit vectors.
      2) Show that u_1 and u_2 are orthogonal.