Do the best that you can to respond to the following problems. It is not required that you respond to every problem, and, in fact, it is much better to provide complete and clearly explained responses to a subset of the given problems than to provide hurried and incomplete responses to all of them. When a complete response is not possible you are encouraged to clearly explain your partial progress and/or your ideas for what a solution might look like.

1. Examples: Provide an example or state that no such example exists. One of your responses requires full justification.
   (a) A function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous at at least one point, but also satisfies the Intermediate Value Property.
   (b) A compact and completely disconnected set in $\mathbb{R}$ with Lebesgue measure 0.
   (c) A Riemann integrable function on $\mathbb{R}$ with an uncountable set of discontinuities.
   (d) A differentiable function $f : [0, 1] \to \mathbb{R}^2$ such that $f(1) - f(0) \neq Df(c)$ for all $c \in (0, 1)$.
   (e) Choose one of your responses above and provide justification.
2. Prove the inequality
$$\liminf_{n\to\infty} \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf_{n\to\infty} |s_n|^{\frac{1}{n}}.$$
3. Find simple conditions on α and β such that if $f_0 = 1, f_1 = 1$, and $f_{n+2} = \alpha f_n + \beta f_{n+1}$, then $a_n := \frac{f_{n+1}}{f_n}$ converges. Your chosen conditions do not need to be optimal, but they should include the case where $\alpha = 1 = \beta$, which is where we get the classic result that the sequence of ratios of the Fibonacci sequence converge to the Golden Ratio.
4. Show that the Divergence Theorem is equivalent to Green's Theorem on suitable domains in $\mathbb{R}^2$.

5. Let $||\cdot||_a$ and $||\cdot||_b$ be two norms on $\mathbb{R}^N$. Recall that a norm, $||\cdot||$, satisfies

   (a) $||x|| \geq 0$ for all x and $||x|| = 0$ only if and only if $x = 0$,

   (b) $||cx|| = |c|||x||$ for all $c \in \mathbb{R}$ and $x \in \mathbb{R}^N$,

   (c) $||x+y|| \leq ||x|| + ||y||$ for all $x, y \in \mathbb{R}^N$.

   Show that these norms are equivalent, i.e. show that there are positive constants c_1 and c_2 such that

   $$c_1||x||_a \leq ||x||_b \leq c_2||x||_a, \text{ for all } x \in \mathbb{R}^N.$$

6. Let $f : \mathbb{R}^N \to \mathbb{R}$ be a differentiable function that satisfies the *Palais-Smale condition*, i.e. if (x_n) is a sequence in $\mathbb{R}^N$ such that $(f(x_n))$ is bounded in $\mathbb{R}$ and $(\nabla f(x_n))$ converges to 0 in $\mathbb{R}^N$, then (x_n) has a converging subsequence. Show that if f is bounded below and satisfies the Palais-Smale condition then f must achieve an absolute minimum.

7. Let A be an $N \times N$ matrix with real entries and let I represent the $N \times N$ identity matrix. Show that there is an $\epsilon > 0$ small enough so that if $||A - I||_2 < \epsilon$ then A has a unique square root in an appropriate neighborhood of A. Note that $||\cdot||_2$ is the standard Euclidean norm on $\mathbb{R}^{N^2}$.

8. Assume that (f_n) is a sequence of continuous functions on $[0,1]$ such that

   (a) There is a countable dense subset $D \subset [0,1]$ such that $\lim_{n\to\infty} f_n(x)$ exists for all $x \in D$, and

   (b) there is a $K > 0$, independent of n, such that $|f_n(x) - f_n(y)| \leq K|x - y|$ for all $x, y \in [0,1]$.

   Show that (f_n) converges uniformly to a continuous function on $[0,1]$.

9. Assume that (g_n) is a sequence of continuous functions on $\mathbb{R}$ such that

   (a) $g_n(x) \geq 0$ for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$,

   (b) $\int_{\mathbb{R}} g_n(x) = 1$ for all $n \in \mathbb{N}$,and

   (c) $\lim_{n\to\infty} \int_{[-\delta,\delta]^c} g_n(x)dx = 0$ for all $\delta > 0$. (Note: For $A \subset \mathbb{R}$ A^c represents the complement of the set A.)

   Assume that $f : \mathbb{R} \to \infty$ is a continuous function which is 0 outside of $[0, 1]$. Let

   $$f_n(x) := \int_{\mathbb{R}} g_n(y)f(x-y)dy = \int_{\mathbb{R}} g_n(x-y)f(y)dy.$$

   Show that

   (a) (f_n) converges uniformly to f.

   (b) If g_n is a polynomial on $[-2, 2]$, then f_n is a polynomial on $[0, 1]$.

10. Consider Lebesgue integrals on $[0, 1]$.

    (a) State Fatou's Lemma and state Lebesgue's Dominated Convergence Theorem.

    (b) Use one to prove the other. Either direction is fine.