Swarthmore Honors Exam 2013: Probability

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Instructions: This is a 3-hour closed-book, closed-note exam. You may use a calculator that does not do algebra or calculus. Show your work and explain your reasoning. The last page contains a table of important distributions.

1. A deck of cards is shuffled well. The cards are dealt one by one, until the first time an Ace appears.

(a) Find the probability that no Kings, Queens, or Jacks appear before the first Ace.

(b) Find the probability that exactly one King, exactly one Queen, and exactly one Jack appear (in any order) before the first Ace.

2. There will be $X \sim \text{Pois}(\lambda)$ courses offered at a certain school next year.

(a) Find the expected number of choices of 4 courses (in terms of λ, fully simplified), assuming that simultaneous enrollment is allowed if there are time conflicts.

(b) Now suppose that simultaneous enrollment is not allowed, and that there are only four possible time slots: 10 am, 11:30 am, 1 pm, 2:30 pm (each course meets Tuesday-Thursday for an hour and a half, starting at one of these times). Rather than trying to avoid major conflicts, the school schedules the courses completely randomly: after the list of courses for next year is determined, they randomly get assigned to time slots, independently and with probability 1/4 for each time slot.

Let X_{am} and X_{pm} be the number of morning and afternoon courses for next year, respectively (where "morning" means starting before noon). Find the joint PMF of X_{am} and X_{pm}, i.e., find $P(X_{\text{am}} = a, X_{\text{pm}} = b)$ for all a, b.

3. You are given an amazing opportunity to bid on a mystery box containing a mystery prize! The value of the prize is completely unknown, except that it is worth at least nothing, and at most a million dollars. So the true value V of the prize is considered to be Uniform on [0,1] (measured in millions of dollars).

You can choose to bid any amount b (in millions of dollars). You have the chance to get the prize for considerably less than it is worth, but you could also lose money if you bid too much. Specifically, if $b < \frac{2}{3}V$, then the bid is rejected and nothing is gained or lost. If $b \geq \frac{2}{3}V$, then the bid is accepted and your net payoff is $V - b$ (since you pay b to get a prize worth V). What is your optimal bid b (to maximize the expected payoff)?

4. Let U_1, U_2, U_3 be i.i.d. $\sim \text{Unif}(0,1)$. Find $P(U_1 U_2 < U_3^2)$.

5. Let $X_1, X_2, X_3, \dots, X_{10}$ be the total number of inches of rain in Boston in October of 2013, 2014, 2015, …, 2022, respectively, with these r.v.s independent $\mathcal{N}(\mu, \sigma^2)$. (Of course, rainfall can't be negative, but μ and σ are such that it is extremely likely that all the X_j's are positive.) We say that a *record value* is set in a certain year if the rainfall is greater than all the previous years (going back to 2013; so by definition, a record is always set in the first year, 2013).

(a) On average, how many of these 10 years will set record values? (Your answer can be a sum but the terms should be simple.)

(b) Determine whether the event that the year 2020 sets a record is independent of the event that the year 2021 sets a record.

(c) Find the probability that the October 2020 rainfall will be more than double the October 2021 rainfall in Boston, in terms of the standard Normal CDF Φ.

6. Suppose that stars are distributed in space according to a 3-dimensional Poisson process with parameter λ (for simplicity, take space to be $\mathbb{R}^3$ without worrying about whether the universe is finite or infinite). This means that the number of stars in any region of space is Poisson with parameter λ times the volume of the region, and the numbers of stars in any two disjoint regions of space are independent.

(a) You are at the origin. Find the distribution of the (Euclidean) distance from you to the nearest star (give the CDF or the PDF).

(b) Let S be a bounded region of space, which has been partitioned into disjoint subregions $S_1, \dots, S_k$. What is the (conditional) joint PMF of the star counts for each S_j, given that there are exactly n stars in S?

Table of Important Distributions

Let $0 < p < 1$ and $q = 1 - p$.

Name	Param.	PMF or PDF	Mean	Variance
Bernoulli	p	$P(X=1) = p, P(X=0) = q$	p	pq
Binomial	n, p	$\binom{n}{k} p^k q^{n-k}$, for $k \in \{0, 1, \dots, n\}$	np	npq
FS	p	pq^{k-1}, for $k \in \{1, 2, \dots\}$	$1/p$	q/p^2
Geom	p	pq^k, for $k \in \{0, 1, 2, \dots\}$	q/p	q/p^2
NBinom	r, p	$\binom{r+n-1}{r-1} p^r q^n, n \in \{0, 1, 2, \dots\}$	rq/p	rq/p^2
HGeom	w, b, n	$\frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$, for $k \in \{0, 1, \dots, n\}$	$\mu = \frac{nw}{w+b}$	$(\frac{w+b-n}{w+b-1}) n \frac{\mu}{n} (1 - \frac{\mu}{n})$
Poisson	λ	$\frac{e^{-\lambda}\lambda^k}{k!}$, for $k \in \{0, 1, 2, \dots\}$	λ	λ
Uniform	$a < b$	$\frac{1}{b-a}$, for $x \in (a, b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Normal	μ, σ^2	$\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$	μ	σ^2
Expo	λ	$\lambda e^{-\lambda x}$, for $x > 0$	$1/\lambda$	$1/\lambda^2$
Gamma	a, λ	$\Gamma(a)^{-1}(\lambda x)^a e^{-\lambda x} x^{-1}$, for $x > 0$	a/λ	a/λ^2
Beta	a, b	$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$, for $0 < x < 1$	$\mu = \frac{a}{a+b}$	$\frac{\mu(1-\mu)}{a+b+1}$
χ^2	n	$\frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}$, for $x > 0$	n	$2n$
Student-t	n	$\frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)} (1 + x^2/n)^{-(n+1)/2}$	0 if $n > 1$	$\frac{n}{n-2}$ if $n > 2$