Swarthmore College Combinatorial Optimization and Probability Honors Exam - Friday May 6, 1994

Instructions: There are two parts, Combinatorial Optimization (Game Theory) and Probability. Spend half of the time on each part.

Answer the questions carefully, showing necessary work. Careful answers to most of (parts of) the questions are more important than careless answers to all of them.

Part I: Combinatorial Optimization (Game Theory)

1. In the game of nim, there are several piles of sticks. Two players take alternate moves. On a player's move, he or she selects one of the piles and removes at least one stick from that pile (i.e., he or she selects a pile and removes anywhere from one stick to all the sticks in that pile). The last player to remove a stick wins. An example of a game is shown below, with the diagram indicating the piles of sticks before the indicated players move.

$$\text{I: }\begin{matrix}1\\11\\111\end{matrix} \Rightarrow \text{II: }\begin{matrix}1\\ \\111\end{matrix} \Rightarrow \text{I: }\begin{matrix}1\\ \\1\end{matrix} \Rightarrow \text{II: }\begin{matrix}1\\ \\ \end{matrix} \qquad \text{(II wins by removing the last stick).}$$

(a) Draw a game tree for the initial configuration $\begin{matrix}1\\11\end{matrix}$. Which player has a winning strategy?

(b) Prove that for any initial configuration of the game of nim one of the players has a winning strategy.

(c) Describe the winning strategy for the game $\begin{matrix}1\\1\\11\end{matrix}$.

Describe in general the winning strategy for the game with k piles with one stick each and 1 pile with two sticks and prove that your description is correct.

2. Use induction on a game tree to explain why the game of chess has a value. That is, either white can always win, black can always win, or a draw can always be forced. (Assume that the game is finite; i.e., a sequence of moves repeated three times results in a draw etc.) You do not need to give a formal proof here, just an explanation. Given that chess has a value, why does it remain an interesting game? Is it reasonable to expect that the value of the game will be determined soon using a fast supercomputer?

3. Give an example of a 2×2 game in normal (matrix) form with no pure strategy Nash equilibrium, and no dominant strategy. Explain why this is the case for your example.

What is the mixed strategy solution to your example?

If you can not give an example as above, give two examples, one with no pure strategy Nash equilibrium and one with no dominant strategy.

4. Consider the matrix form of a zero sum game with strategies $a_1, a_2, \ldots a_m$ for player I and strategies $b_1, b_2, \ldots b_n$ for player II. Assume that both (a_i, b_j) and (a_r, b_s) are pure strategy saddlepoints. (That is, minimax for player I and maximin for player II.) Prove that (a_i, b_s) and (a_s, b_r) are also saddlepoints.

5. Consider the zero sum game given by the matrix

1	-1	-2
-1	1	1
2	-1	-1

with entries indicating the payoff to the row player.

(a) Use dominated strategies to reduce this to a 2×2 game.

(b) Use the minimax theorem to verify that the mixed strategies $(0, 3/5, 2/5)$ for the row player and $(2/5, 3/5, 0)$ for the column player are optimal for both. (Use the original matrix, not that obtained in part (a).)

(c) Write down (but do not solve) a linear programming problem to determine the maximin strategy for the row player (again use the matrix from part (a)). Explain your reasoning. Do the same for the column player. How are these related in linear programming terminology?

(d) Describe as carefully as you can (using whatever notation you prefer) the connection between linear programming duality and von Neumann's minimax theorem for matrix games. If time permits, sketch proofs.

6. Consider a conference committee consisting of three senators x, y and z and three members of the House of Representatives a, b and c. A measure passes this committee if and only if it receives the support of at least two senators and at least two representatives.

(a) Determine the characteristic function of this game.

(b) Show that this game is not a weighted majority game. That is, show that we cannot find votes $v(x), v(y), v(z), v(a), v(b)$ and $v(c)$ and a quota Q such that a measure passes if and only if the sum of the votes in favor is a least Q.

(c) Describe how to generalize this to any bicameral legislature with two chambers having, respectively m and n members and passage requiring a majority in each chamber.

(d) Determine the Shapley value and Banzhaf index of the game with three senators and three representatives.

(e) Determine the Shapely value and Banzhaf index of the game with three senators and five representatives.

(f) Write down expressions (you may not be able to evaluate them) for the Shapely value and Banzhaf index of the game with m senators and n representatives.

7. Consider the following three person game (an adaptation due to Shapley and Shubik of a game described by von Neumann and Morgenstern). A given tract of land is currently worth \$100,000 to player I, its owner, who uses it for agricultural purposes. A prospective industrial user, player II, considers the land worth \$200,000 and a prospective subdivider, player III, considers it worth \$300,000. There are no other prospective buyers.
Thus, the characteristic function is
$v(\{1\}) = \$100,000; v(\{2\}) = v(\{3\}) = v(\{2,3\}) = \$0;$
$v(\{1,2\}) = \$200,000; v(\{1,3\}) = v(\{1,2,3\}) = \$300,000.$

(a) Determine the core of this game. Give an economic interpretation, if possible, to the set of imputations in the core.

(b) Verify that the core determined in part (a) is not a stable set.

(c) Verify that $B = \{(x_1, x_2, x_3) : \$100,000 \leq x_1 \leq \$200,000, x_2 = \$100,000 - \frac{1}{2}x_1, x_3 = \$200,000 - \frac{1}{2}x_1\}$ is a stable set. Give an economic interpretation.

(d) Find another (distinct) stable set.

(e) Determine the $(0,1)$ normalization of the game. (You may wish to do this first as an aid in answering the previous parts.)

Part II: Probability

1. Write down an expression to evaluate the probability that in a room with 40 people, at least two will share a birthday. Write down the expression for n people in the room.

2. In a class with 3 students the instructor hands back exams randomly. Use the principle of inclusion-exclusion to determine the probability that none of the students gets their own paper back. Do the same for a class of size 4. Finally, give a general expression for the probability that none of the students in a class of size n gets their own exam back when the exams are randomly returned.

3. What is more probable: to toss 6 at least once out of four tosses of a fair die or to toss 12 at least once out of 24 tosses of two dice?

4. If $f(x) = 1/2, -1 \leq x \leq 1$, $f(x) = 0$ elsewhere is the probability density function of a random variable X, find the probability density function of $Y = X^2$.

5. What is a random variable? Give as precise a definition as you can. Give also an informal description; one that you might give to a friend who has little background in mathematics.

6. Let $X_1, X_2, \ldots X_n$ be independent random variables from a distribution with mean μ and variance σ^2. Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.

(a) Show that $\mathrm{E}(\overline{X}) = \mu$.

(b) Show that the variance $\mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}$.

(c) Recall that Chebyshev's inequality (in one version) states that for a random variable with mean μ and variance σ^2, $P(|X - \mu| < k\sigma) \geq 1 - \frac{1}{k^2}$. Use Chebychev's inequality to prove one version of the law of large numbers. That is, prove that for every $\epsilon > 0$, $\lim_{n\to\infty} P(|\overline{X} - \mu| < \epsilon) = 1$.

(d) Give an informal explanation of the result of part (c).

7. If A and B are events on a sample space, recall that the conditional probability of A given B is given by $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

(a) Prove the following simple version of Bayes' formula for events C and D,

$$P(C|D) = \frac{P(D|C)P(C)}{P(D)}.$$

(b) Suppose $C_1, C_2, \ldots C_n$ are mutually disjoint events with nonzero probability and suppose that C is a subset of the union of $C_1, C_2, \ldots C_n$. Show that $P(C) = P(C_1)P(C|C_1) + \cdots + P(C_n)P(C|C_n)$.

(c) Prove Bayes' formula, $P(C_i|C) = \dfrac{P(C|C_i)P(C_i)}{P(C_1)P(C|C_1) + \cdots + P(C_n)P(C|C_n)}$.

(d) A certain test for a disease is always positive if you have the disease. There is also a small chance of a false positive; if you do not have the disease, the probability of a positive result is 1 in 1000 (i.e., 0.001). Suppose also that the incidence of the disease in the general population is 1 in a million; the probability of having the disease is 1/1000000. Assume that you are randomly selected to take the test for the disease and that the test comes back positive. What is the probability that you have the disease?

(e) Suppose that you go to your doctor and ask for the test described in part (d). The test comes back positive. What is the probability that you have the disease? What different assumptions are being made here? Why is this important in decisions to actually use the test?

8. (a) Consider a Bernoulli trial with a probability p of success. Let $b(k; n, p)$ be the probability that n independent trials result in k successes and $n - k$ failures. Determine $b(k; n, p)$.

(b) Let X be a random variable with density $f(x) = \binom{n}{k} p^x (1-p)^{n-x}$ for $x = 0, 1, 2, \ldots, n$ and $f(x) = 0$ elsewhere. Determine the moment generating function of f and use this to determine the mean and variance of X.