SWARTHMORE COLLEGE
DEPARTMENT OF MATHEMATICS AND STATISTICS
HONORS EXAMINATION IN GEOMETRY, 2013

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Instructions: Do as many problems or parts of problems or special cases of problems as you can. Justify all answers. Feel free to mention multiple (substantially different) methods for solving a single problem. You may quote any standard result as long as that result is not essentially what you are asked to prove.

(1) If C is a closed curve contained inside a disk of radius r in $\mathbb{R}^2$, prove that there exists a point $p \in C$ where the curvature satisfies $|\kappa_g(p)| \geq \frac{1}{r}$. What analogous statement is true if C is an n-dimensional compact Riemannian manifold in $\mathbb{R}^{n+1}$ contained inside an n-dimensional sphere of radius r? Is this result still true if C is a compact Riemannian manifold of *any* dimension in $\mathbb{R}^{n+1}$ contained inside an n-dimensional sphere of radius r?

(2) Prove that S^n (the n-dimensional sphere of radius 1) has constant sectional curvature equal to 1.

(3) Consider the sphere and the cylinder:

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$
$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}.$$

Let $f : (S^2 - \{(0, 0, 1), (0, 0, -1)\}) \to C$ denote the function that sends each point of the domain to the closest point in C. Prove that f is "equiareal" which means that it takes any region of the domain to a region of the same area in C.

(4) Let $M^2 \subset \mathbb{R}^3$ be a ruled surface. This means that for all $p \in M^2$ there exists a line in $\mathbb{R}^3$ through p which is entirely contained in M^2. Explain why the Gauss curvature of M^2 is nonpositive.

(5) Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2 - 1\}$, and let C be the circle along which M intersects the xy-plane.
   (a) The parallel transport once around C has the effect of rotating the tangent space $T_{(1,0,0)}M$ by what angle?
   (b) What is the integral of the Gauss curvature over the region of M that is bounded by C?

(6) Let M be a Riemmanian manifold, and let $f : M \to \mathbb{R}$ be a smooth function. Define grad$f$ to be the unique vector field on M such that for all $p \in M$ and all $X \in T_pM$, $\langle (\text{grad} f)(p), X\rangle = df_p(X)$. If grad$f$ has constant norm on M, prove that all integrals curves of grad$f$ are minimizing geodesics.

(7) Let M be a Riemannian manifold. A "ray" in M is define as a geodesic $\gamma : [0, \infty) \to M$ which is minimizing between any pair of points of its image. Suppose that $p \in M$ and $\{V_n\} \to V$ is a convergent sequence of vectors in T_pM. Suppose that for each n, the geodesic in the direction of V_n is a ray. Prove that the geodesic in the direction of their limit, V, is a ray.

(8) A Riemannian manifold M is called "homogeneous" if for every pair $p, q \in M$ there exists an isometry $f : M \to M$ such that $f(p) = q$. A Riemannian manifold M is called a "symmetric space" if for every $p \in M$ there exists an isometry $f : M \to M$ such that $f(p) = p$ and $df_p : T_pM \to T_pM$ equals the antipodal map $V \mapsto -V$. Prove that every complete symmetric space is homogeneous. What examples of symmetric and homogeneous spaces can you think of?

(9) What can you conclude about a surface $M^2 \subset \mathbb{R}^3$ for which the image of the Gauss map is contained in a great circle of S^2?

(10) Let $S^n(r)$ denote the n-dimensional sphere of points in $\mathbb{R}^{n+1}$ at distance r from the origin. Let $m, n \geq 1$ be integers and let $s, r > 0$ be real numbers. The product manifold, $M = S^m(r) \times S^n(s)$, can be identified with the following subset of $\mathbb{R}^{m+n+2}$:

$$M = \{(p, q) \in \mathbb{R}^{m+1} \oplus \mathbb{R}^{n+1} \cong \mathbb{R}^{m+n+2} \mid p \in S^m(r), q \in S^n(s)\},$$

and therefore M inherits a natural "product" metric from the ambient Euclidean space. Under what conditions on $\{m, n, s, r\}$ (if any) will M have...

(a) ...positive sectional curvature?
(b) ...positive Ricci curvature?
(c) ...positive scalar curvature?
(d) ...constant Ricci curvature?