Swarthmore College
Department of Mathematics and Statistics
Honors Examination in Geometry 2012

Instructions: Do 6 of the following 12 problems as thoroughly as you can. Try to include at least one problem from each of the four parts of the exam. You may use without proof the basic theorems that you have learned, but be sure to state them carefully. You may also use a result from one problem in solving another, even if you didn't solve the first problem. If you have lots of time left over, feel free to solve other problems.

Notation

The following conventions for notation have been followed:

$\mathbb{R}$ = the set (group, field) of real numbers.
$\mathbb{C}$ = the set (group, field) of complex numbers.
S^{n-1} = the unit sphere in $\mathbb{R}^n$.

Synthetic and Analytic Geometry

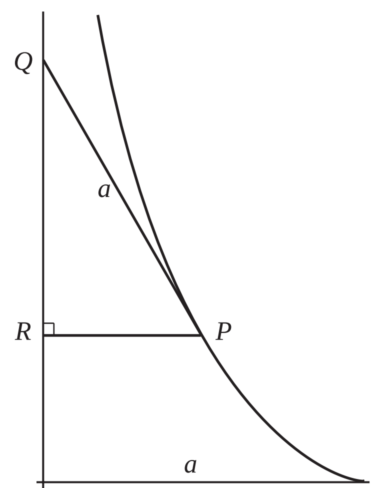
1. There are three families of lines found in each of hyperbolic, Euclidean, and elliptic geometry called *pencils of lines*: there is $\mathcal{P}_P$, the collection of all lines passing through the point P; $\mathcal{P}^l_{\|}$, the set of all lines parallel to a given line l; and $\mathcal{P}^l_{\perp}$, the collection of all lines perpendicular to a given line l. Show that the perpendicular bisectors of the sides of a triangle always fall into one of these pencils, and show that all pencils are realized in the hyperbolic plane. Show that the pencil of parallels is a pencil of perpendiculars if and only the Parallel Postulate holds.

2. The Euclidean plane is described by the axioms and definitions of Euclid, and by the axiom scheme of Hilbert. Compare the two approaches to the geometry of the plane, their goals, their methods, and feature their differences in a short essay of a half-page to a page.

3. Show that a line in the plane makes an angle $\psi(R)$ with the circle of radius R centered at the origin which satisfies $R\cos(\psi(R))$ = a constant, that is, the product $R\cos(\psi(R))$ does not depend on R.

4. Let $\alpha\colon \mathbb{R} \to \mathbb{R}^3$ be given by $\alpha(t) = (\cosh(t), \sinh(t), t)$, a space curve. This parametrization is nice because the arc length turns out nicely and a unit speed parametrization is easy to obtain. Determine the arc length parametrization and use it to compute the curvature of $\alpha(t)$. (Useful fact: $\operatorname{arccosh}(s) = \ln(s + \sqrt{s^2 - 1})$.)

5. An important curve introduced in the 17th century is the *tractrix*, the curve along which a small object moves when pulled on a horizontal plane with a piece of thread by a puller which moves along a straight line.



The tractrix

A unit speed parametrization of the tractrix curve is given by

$$\Theta(s) = \left(e^{-s}, \ln(e^s + \sqrt{e^{2s} - 1}) - \frac{\sqrt{e^{2s} - 1}}{e^2}\right).$$

The osculating circle associated to a curve is the circle tangent to the curve at each point with radius given by the reciprocal of the curvature of the curve at the point. The center of the osculating circle lies along the normal to the curve at the point and the collection of all such centers forms a curve called the *curve of centers of curvature*. (An evolute of a curve is obtained from the curve of centers of curvature). Determine a parametrization of the curve of centers of curvature of the tractrix and show that it is a catenary curve (that is, isometric to $y = \cosh(t)$).

6. Prove the theorem of Tinseau, that the orthogonal projection of a space curve onto a plane has a point of inflection (a point where $\kappa(s) = 0$) if the plane is perpendicular to the osculating plane of the curve (the plane spanned by $T(s)$ and $N(s)$).

Geometry on surfaces

7. Euclid's posulates can be interpreted as statements about a general surface. What properties of surfaces need be satisfied in order to realize each of Euclid's postulates? Explain.

8. For a surface S in $\mathbb{R}^3$, the area formula from multivariable calculus can be used with geodesic polar coordinates to compute the area of a geodesic circle centered at $p \in S$ of radius r, denoted $\text{area}_p(r)$. Use these tools to prove a theorem of Diguet that computes the Gaussian curvature:

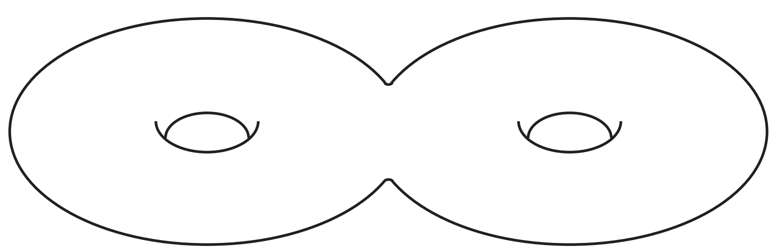
$$K(p) = \lim_{r \to 0^+} \frac{12(\pi r^2 - \text{area}_p(r))}{\pi r^4}.$$

9. The Beltrami disk is the subset of $\mathbb{R}^2$ given by $\mathbb{D} = \{(u, v) \mid u^2 + v^2 < 1\}$ together with the metric

$$ds^2 = \frac{(1 - v^2)\, du^2 + 2uv\, du\, dv + (1 - u^2)\, dv^2}{(1 - u^- v^2)^2}.$$

In this abstract surface, straight line segments are geodesics. Measuring from the origin, determine the distance ρ between $(0, 0)$ and $(0, r)$ for $r > 0$. Using lines in $(\mathbb{D}, ds^2)$, prove the relation $\cos(|Pi(\rho)) = \tanh(\rho)$, where $\Pi(\rho)$ is the angle of parallelism for the length ρ.

10. The compact surface F of genus 2 is homeomorphic to an abstract surface F' of constant curvature. Determine $\int_F K\, dS$ from the Euler characteristic of F. What kind of geometry do we find on F'?



11. The Christoffel symbols (of the first kind) are defined in local coordinates by

$$\Gamma_{ij,k} = \frac{1}{2}\left(-\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i}\right) \quad \text{or } \Gamma_{ij,k} = \left\langle \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right\rangle.$$

Determine the change of variables formula for the Christoffel symbols and explain why the $\Gamma_{ij,k}$ do **not** constitute a tensor.

12. The torus in $\mathbb{R}^4$ may be expressed as the algebraic set

$$T = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 = 1 = z^2 + w^2\}.$$

As a subset of $\mathbb{R}^4$, the torus is a Riemannian manifold. Determine local coordinates on T and the associated apparatus of Christoffel symbols. Compute the RIemannian curvature of the torus with this embedding.