Swarthmore College
Department of Mathematics and Statistics
2009 Honors Examination in Geometry
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This exam covers more material than any single student is expected to know, or will have time to solve. Feel free to skip around. The problems are of variable difficulty, so work the problems that you find easiest first. You are invited to sketch your approach to problems that you don't have time to solve completely, and to solve parts of problems while leaving earlier or later parts unsolved.

[**1**] The map $\gamma : \mathbb{R} \to \mathbb{R}^3$ given by

$$\gamma(t) \;=\; (t, t^2, t^3)$$

defines a curve in $\mathbb{R}^3$. Compute its curvature and torsion.

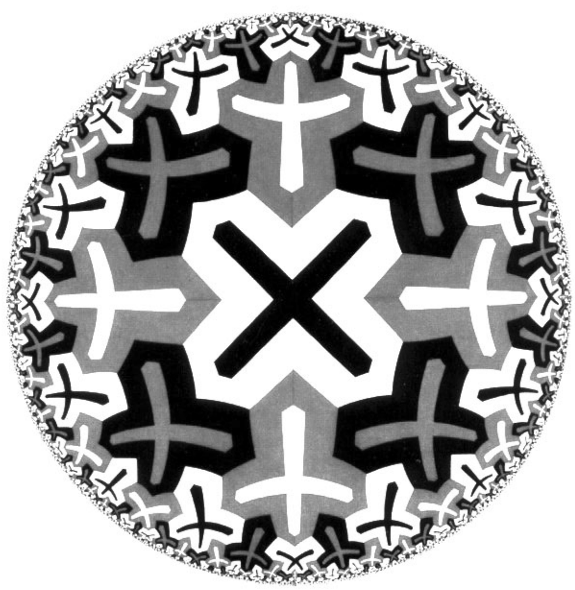
[**2**] (a) Give a parametrization of a torus S embedded in $\mathbb{R}^3$.

(b) Compute the Gauss map of S.

(c) Compute the Gaussian curvature at each point of S. What are the principal curvatures?

(d) Describe the Gauss map of the surface $(x + y + z)^2 - 2(x^2 + x^2 + z^2) = 0$.

[**3**] (a) What aspects of hyperbolic geometry can you illustrate using the following Escher drawing, *Circle Limit II*?



(b) In combination, the uniformization theorem for surfaces and the Gauss-Bonnet theorem tells us that a closed surface of genus > 1 can be given a Riemannian metric of constant negative curvature. One way to form a genus 2 surface is by identifying sides of an octagon. Does the above tiling bear any relation to such a surface? Why or why not?

[4] A *monomial ideal* is an ideal in a polynomial ring $k[x_1, \dots, x_n]$ generated by single terms. For example, $I = (x^2, xy, y^2)$ is a monomial ideal in the polynomial ring $\mathbb{C}[x, y, z]$.

(a) Let I be a monomial ideal in the two variable polynomial ring $k[x, y]$. Show that I is finitely generated, without using the Hilbert basis theorem.

(b) Can you extend your proof to monomial ideals $I \subset k[x, y, z]$? (The general case is known as Dickson's lemma. It can be used to give an alternate proof of the Hilbert basis theorem.)

(c) Now suppose that $I \subset k[x_1, \dots, x_n]$ is a *radical* monomial ideal. Give a bound on the number of monomials needed to generate I.

[5] (a) State the *Nullstellensatz*.

(b) Give counterexamples over the real numbers $\mathbb{R}$, the rational numbers $\mathbb{Q}$, and the finite field with three elements $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$.

(c) Algebraic geometers prefer to work over fields where the Nullstellensatz holds, and they prefer to work in projective space. Why?

[6] (a) The *Veronese surface* is the image of the map $\alpha : \mathbb{P}^2 \to \mathbb{P}^5$ where

$$\alpha(x, y, z) = (x^2, xy, xz, y^2, yz, z^2)$$

Show that the Veronese surface is nonsingular.

(b) What is the tangent plane at each point $\alpha(x, y, z)$ of the Veronese surface?

(c) Each tangent plane corresponds to a point in the Grassmanian $G(3,6)$ of projective planes in $\mathbb{P}^5$. Describe the map

$$\alpha : \mathbb{P}^2 \to G(3,6)$$

which takes each point (x, y, z) of $\mathbb{P}^2$ to the tangent plane of $\alpha(x, y, z)$.

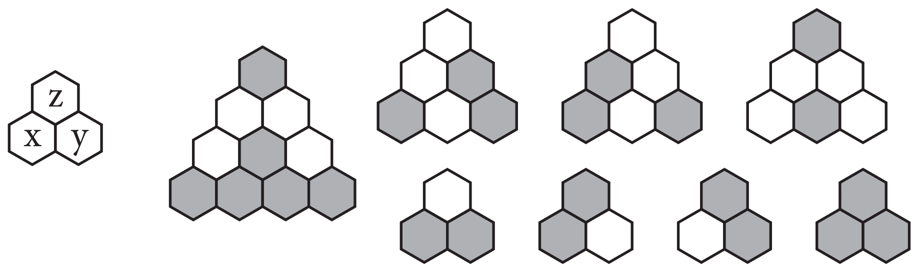
[7] (a) Describe the zero locus $X \subset \mathbb{A}^2_{\mathbb{C}}$ of the equation $xy^2 - x^4 = 0$ in the affine plane over the complex numbers. Is X irreducible? Are the component(s) of X nonsingular? Describe the strict transform $\widetilde{X}$ of X after blowing up the origin. ($\widetilde{X}$ is "what happens to" X after blowing up.)

(b) Let Y be the curve defined by $y^2 - x^d = 0$ for d odd. What is the singular locus of Y? Can you use blowing up to transform Y into a nonsingular curve?

(c) The mathematician John Nash studied a variation on blowing up, associating each smooth point of a variety X with its tangent space, and taking the closure in hopes of smoothing out the singular locus. Can you make this idea work for $y^2 - x^d = 0$?

[**8**] (a) The cubic equation $(x + y)^3 + xyz + z^3 = 0$ has four nontrivial integer solutions mod 2. What are they?

(b) This equation defines a cubic curve $X \subset \mathbb{P}^2$, and the above solutions are the four rational points of X over the finite field with two elements $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Show that X is nonsingular at each of these points.



(c) The above illustration represents the cubic equation, its partial derivatives, and the solutions as diagrams shading the terms or entries that are nonzero. Can you devise rules for carrying out the calculations in (a) and (b) visually?

(d) The usual group law on elliptic curves can be used here to construct a group of order four. There are two possibilities: $C_2 \times C_2$, or C_4. Which group do we get? Can the other group also arise?

[**9**] (a) Spherical distance is angle. Hyperbolic distance is sometimes described as imaginary angle. Certainly, the identity

$$\cosh(x) \;=\; \cos(ix)$$

suggests this. Can you reconcile any formulas from spherical and hyperbolic geometry using this idea?

(b) Work with the unit 2-sphere, and let

$$X \;=\; \{\, (a^2, b^2, c^2) \;\mid\; \text{there exists a triangle with side lengths } a, b, c \,\}$$

Does X consist of an entire orthant? Is X unbounded? Give a geometric interpretation for X in a neighborhood of the origin.

(c) Now work with a hyperbolic plane having Gaussian curvature -1, and consider distances to be imaginary angles. Let

$$Y \;=\; \{\, (-a^2, -b^2, -c^2) \;\mid\; \text{there exists a triangle with side lengths } ai, bi, ci \,\}$$

Does Y consist of an entire orthant? Is Y unbounded? Give a geometric interpretation for Y in a neighborhood of the origin.

(d) Algebraic geometers like to blow things up. (Hey, when you have a hammer, everything looks like a nail.) What can you say about the above picture after blowing up the origin?